Developing Mathematical Understandings

in the Upper Grades

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PREFACE

TEACHERS of mathematics everywhere are concerned with ways of improving the mathematics program. The tremendous increase in enrollments above the elementary school level in recent years has raised problems which teachers often find exceedingly difficult to deal with effectively. The current promotion trend of "uninterrupted continuity" has complicated the situation greatly because many children leave the elementary school who are seriously deficient in arithmetic skills. As a consequence many changes are being made in the organization of the mathematics curriculum in the upper grades and in methods and materials of instruction so as to adjust the work in mathematics to the wide range in ability of various students.

DEVELOPING MATHEMATICAL UNDERSTANDINGS IN THE UPPER GRADES presents a non-technical discussion of problems related to curriculum and instruction in mathematics at the junior high school level. It stresses the instructional procedures that research has shown not only will make mathematics meaningful to the students but also will insure ability on their part to apply systematic procedures effectively in dealing with the quantitative aspects of social situations of daily life. These themes were developed for elementary arithmetic in an earlier companion volume by two of the authors, *Making Arithmetic Meaningful.*

DEVELOPING MATHEMATICAL UNDERSTANDINGS IN THE UPPER GRADES is written for (1) college students who are preparing to teach mathematics at the junior high school level, (2) classroom teachers of mathematics and related areas, and (3) curriculum specialists. The major problems with which this book deals are those which all mathematics teachers face at the junior high school level. They may be listed as follows:

1. What are the goals of the mathematics program? How should they be formulated?

2. How should the content of the mathematics curriculum be selected and organized? What kinds of instructional units should it include?

3. How can instructional procedures and content be adjusted to the wide range of differences among the students in mental ability, needs, and levels of development?

4. What kinds of experiences and instructional equipment are most likely to develop in students of various levels of ability an understanding of the basic operations in arithmetic and algebra?

5. What testing and diagnostic procedures can be used by teacher and students to determine learning difficulties and deficiencies in meanings, understandings, and skills as a basis of planning review work at the beginning of the year or in connection with new topics being presented?

6. What is the role of problem solving?

7. What special provisions can the mathematics program make for gifted and talented students?

Mathematics teachers often must reteach some of the fundamentals that already should have been mastered. To assist teachers to do this the authors discuss in some detail the essentials of a remedial teaching program in arithmetic. Every mathematics teacher faces the problem of providing a variety of challenging activities for students of all levels of ability.

The authors point out repeatedly that a student learns best and is most likely to remember what he understands. Understanding is developed through a variety of experiences in which the student makes discoveries of relationships and generalizations. The authors have formulated a series of mathematical principles which teachers will find of real value in developing relationships underlying all number operations. The application of these abstract principles is stressed throughout this book.

The authors have brought to bear on the above problems and topics the findings of modern research on curriculum, learning, and instruction. Instead of presenting detailed reviews of the numerous studies, the authors have followed the plan of presenting concise summaries of important findings and then listing principles derived from them.

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Chapter 1

Foundations of the Modern Mathematics Program

In the past few decades there have been many changes in the mathematics program in the upper grades. In this volume, primary emphasis is placed on emerging trends in the mathematics program at the junior high school level, especially in the area of general education. These changes can be identified in the accepted goals of instruction, in adaptations of the mathematics program to meet the emerging needs of the rapidly growing population of our secondary schools, and in modernized instructional procedures used in classrooms. In this chapter we shall deal with the following topics:

a. The purposes of teaching mathematics in the upper grades
b. Mathematics in relation to the special functions of the junior high school
c. Factors affecting the mathematics program.

a. The Purposes of Teaching Mathematics in the Upper Grades

The Goals of Mathematical Instruction

The goal of all instruction in the upper grades, in particular at the junior high school level, is to provide a learning program that is adapted to the needs, ability, and interests of youth emerging into adolescence. This program should be so organized

that it contributes effectively to the advancement of the general aim of the school, namely: the optimum growth and development of each student both as an individual and as an effective participating member of our dynamic democratic society.

The primary objectives of the modern mathematics program, insofar as general education is concerned, are (1) to develop in the learner the ability to utilize mathematical processes skillfully, intelligently, and with insight into their value and efficiency and (2) to provide a rich variety of experiences which will assure the ability of the learners to apply quantitative procedures effectively in dealing with problems and social situations in the affairs of daily life both in and out of school.

The close relation between mathematical procedures and their social applications is reflected by the contents of most modern courses of study and teaching materials. Many excellent instructional units appear in courses of study and guides for teachers in which the strictly mathematical phase of the work in arithmetic, algebra, and geometry is closely related to the social phase of these subjects. These units emphasize the actual application of quantitative procedures in social situations that arise in the affairs of daily life. The practice that frequently prevailed in the past in so many schools of limiting instruction in arithmetic and related courses to the relatively narrow mathematical phase of these subjects is gradually being discarded in favor of a plan which assures a broad, well integrated treatment of the technical as well as the social aspects of mathematics.

An Analysis of the Outcomes of Instruction in Arithmetic

Many of the great developments for promoting and implementing social intercourse are mathematical in nature, including systems of number, efficient computational procedures, the uses of general numbers, and the geometry of design. Other developments are social in nature, such as measurement, money, insurance, taxation, banking, and many other forms of cooperative group activities set up by society to deal with human needs.

The outline on the opposite page presents a detailed analysis of the desirable outcomes of instruction in arithmetic.

1. Outcomes related to the mathematical phase:

   a. An understanding of the structure of the decimal number system and an appreciation of its simplicity and efficiency
   b. The ability to perform computations connected with social situations with reasonable speed and accuracy
   c. The ability to make dependable estimates and close approximations
   d. Resourcefulness and ingenuity in perceiving and dealing with quantitative aspects of situations
   e. Understanding of the technical vocabulary used to express quantitative ideas and relations
   f. Ability to use and to devise formulas, rules of procedure, and methods of bringing out relations
   g. Ability to represent designs and spatial relations by drawings
   h. The ability to arrange numerical data systematically and to interpret information presented in graphic or tabular form.

2. Outcomes related to the social phase:

   a. Understanding of the process of measurement and skill in the use of instruments of precision
   b. Knowledge about the development and social significance of such institutions as money, taxation, banking, standard time, and measurement
   c. Knowledge of the kinds and sources of information essential for intelligent buying and selling and for general economic competence
   d. Understanding of the quantitative vocabulary encountered in reading, in business affairs, and in social relations
   e. Appreciation of the contributions number has made to the development of social cooperation and to science
   f. Ability and disposition to secure and utilize reliable information in dealing with emerging personal and community problems
   g. Ability to rationalize and analyze experience by utilization of quantitative procedures.

The Special Goals of Instruction in Algebra and Geometry

Algebra is an expansion of the science of numbers, that is, a way of working with general numbers represented by letters. The purposes of instruction in algebra insofar as general education is concerned should be (1) to give every one a general idea

of its meaning, (2) to teach definite and useful applications which everyone is likely to meet, (3) to introduce the students to a powerful and stimulating way of thinking about quantitative relationships, and (4) to explore mathematical aptitudes and to stimulate an interest in the continued study of mathematics. (See Chapter 12 for a more detailed discussion.)

The introductory course in algebra has become largely a study of formulas, their derivation and evaluation, the rules they represent, and the ways in which they are used in science and other fields of study, and in the practical affairs of daily life. The study of equations and graphs is a means for carrying on the work with the formula. Intuitive geometry and indirect measurement at the levels of Grades 7 to 9 deal with the geometry of position, form and size, common areas and volumes, scale drawings, the methods of measuring of inaccessible distances, and *numerical trigonometry*.

The chief purpose of this work in algebra and geometry is to extend the student's knowledge of functional mathematics and to explore and expand his interests and aptitudes in this field of learning.

The Cultural Value of Mathematics

The Commission on Post-War Plans after a discussion of the personal uses of mathematics commented as follows on the cultural value of mathematics:

> It wouldn't be fair to you if we ended this discussion of mathematics for personal use without mentioning *enjoyment* and *cultural* values. There is much more in the study of mathematics than the vocational value. Throughout the centuries there have been many persons who have enjoyed mathematics for its own sake. There is a precious cultural value that comes to those who learn to appreciate the contribution of mathematics to civilization—to the progress of man. You are entitled to know (1) that few students if any who are competent in mathematics have ever regretted time spent in learning the subject, (2) that mathematics is an easy subject if well taught and the structure is built carefully step by step, and (3) that there are few subjects which students like better than mathematics, provided it is well taught. So, if you

enjoy mathematics, take no thought of the vocational uses in the tomorrow—sufficient for this day is the *good* thereof.[1]

Contributions of Mathematics to the General Outcomes of All Education

In addition to these special objectives of mathematics, instruction in this field must take into consideration the broader objectives of education to which all curriculum areas contribute. These include on the one hand outcomes related to the development of desirable aspects of the learner's personality, including his interests, attitudes, appreciations, mental health, emotional adjustment, and social traits, even his physical well being. Hence the necessity of a rich, vitalized, well integrated program adapted to the needs, interests, aptitude, and stage of maturity of the various students.

There are also outcomes of a deep societal significance to which the program in mathematics should make valuable contribution. Not only should the students learn about mathematical aspects of the affairs of daily life, but they should also have experience in the cooperative study and solution of vital problems that arise in their own affairs so that they will become increasingly more able to participate effectively in the study and solution of problems in life outside the school. This requires that they participate in cooperative group processes in problem solving and be given the opportunity to develop qualities of leadership in the group process. Thus the students should become sensitive to problems of the community and be able to participate in the creative thinking that is necessary to improve conditions of community life.

These social outcomes affect the content of what is taught and the methods of instruction that are used. The specific abilities that are involved in the study and solution of significant social problems, for instance in consumer education, which are particularly important as far as mathematics is concerned, may be listed as on the following page.

[1] "Second Report of the Commission on Post-War Plans," *The Mathematics Teacher*, 38:195–221.

1. The ability to sense problems and to formulate them clearly and specifically
2. The ability to formulate methods of arriving at the solutions of these problems
3. The ability to sense and to identify by suitable means the kinds of quantitative information inherent in a situation which can then be extracted so as to make it more meaningful
4. The ability to locate, gather, organize, and present essential pertinent information, both social and quantitative in nature
5. The ability to perceive relations between quantitative elements in situations and to present or express them by an appropriate terminology
6. The ability to arrive at correct conclusions and also to demonstrate that they are reasonable by using approximations
7. The ability and disposition to work cooperatively with others in group activities of various kinds.

This broadened conception of educational objectives is an illustration of the impact on education of modern conceptions of the meaning of democracy, and of the great fund of new knowledge about human relations, personality, and mental health.

b. Mathematics in Relation to the Special Functions of the Junior High School

In modern educational literature[2] six major functions of the junior high school are recognized: (1) integration, (2) exploration, (3) guidance, (4) differentiation, (5) socialization, and (6) articulation. We shall consider briefly the relation of mathematics instruction to each of these functions.

Integration

The term "integration" has a number of different connotations. With respect to the curriculum it means the acceptance of

[2] Gruhn, William T. and Douglass, Harl R. *The Modern Junior High School*, Chapter 3. New York: Ronald Press, 1947.
Noar, Gertrude *The Junior High School—Today and Tomorrow*, Chapter 1. New York: Prentice-Hall, Inc., 1953.

a common philosophy by the faculty of a school which permits teachers to build a program within which basic concepts and skills can be established and made functional, habits of thought and action can be grounded, and the development of basic learnings assured.

Integration of learnings is another underlying point of view. Within the modern school, units of work which cut across subject matter lines often replace the compartmentalized program with its departmentalized program. Instruction in mathematics should be so organized that it makes its unique contributions to all areas of the curriculum. Students should be given the opportunity to see how the various branches of mathematics function in the social, civic, and industrial life of the community.

Integration of personality is a third aspect of this concept. A major concern of instruction in mathematics should be the factors in its program that contribute to mental and social health and personality adjustment. Interest, successful learning, and a feeling of belonging to the group are a few of the most significant factors here involved.

Exploration

The learner in the junior high school should have the opportunity to become familiar with all essential areas of mathematics and to explore his aptitude for this line of study. On this basis his future work in mathematics can be planned. He should also have the opportunity to explore the uses of mathematics in the social environment, including the working world, job opportunities, social institutions, and the practical arts.

Guidance

The placement of the individual in courses of mathematics should be based on information about his mental ability, his aptitude for the subject, his needs, his interests, and his plans for the future. Guidance in learning mathematics requires the study by all teachers, not only the homeroom teacher, of the personality of the learner. The mathematics teacher also should

consider continuously ways of adapting curriculum, instruction, and materials to the student's needs, his learning ability, and the level of development of his concepts. The possibility of stimulating more interest in mathematics through participation in co-curricular activities, such as mathematics clubs, science clubs, and the like, should be carefully considered.

Differentiation

The steps to be taken to deal with the problem of providing for individual differences in ability, needs, interests, and level of development involve many possible adjustments of curriculum, methods, and materials. So-called plans of "homogeneous" grouping and "ability" grouping should not be regarded as a solution to these problems. Nor are double- and triple-track plans of curriculum organization that are being developed in some schools effective ways of adjusting instruction to individual differences. However, these procedures do provide differential programs for students of different levels of ability in mathematics.

A game device for drill may add interest.

John Mills School, Elmwood Park, Illinois

Today differentiation is recognized as a function of the classroom in which the teacher is at work. The teacher must see to it that the experiences of the individual pupil are worthwhile to him and offer a challenge that he has the capacity to meet. Thus when a class whether homogeneous or heterogeneous in ability is dealing as a group with some problem or studying some topic of concern to them, each individual regardless of his level of ability can be given the opportunity to contribute to the group purpose in terms of his interests, needs, talents, and background. At the same time instruction related to the learning of the various mathematical operations and skills as such can to a large extent be individualized, since the rate of learning them varies considerably among the students and their learning difficulties vary from individual to individual.

Socialization

Instruction in mathematics can be so organized that the learner has experience in the ways in which the dynamic processes of civilized society operate. Through the study and solution of significant problems he should learn about the group process and how it proceeds in order to acquire the understandings, skills, and appreciations that are needed for successful effective participation in the affairs of our democratic society. Socialization is predominantly related to method of instruction, although the contents of the mathematics curriculum offer abundant opportunity for children to study the social institutions through which a cooperative democratic way of life functions, such as credit unions, insurance, taxation, banking, voting, and the like. The experience unit is admirably adapted for these activities.

Articulation

The traditional concept of articulation involves the steps taken to reduce the difficulty of transition from one level of the school to another, for example, from the elementary school to the junior high school, or from the junior high school to the senior high school, or from one school to another. When all of the schools

of a community have a common philosophy, the schools can make efforts to plan a continuous mathematics program from the kindergarten through the secondary school.

The school should also take steps to help the learners to accept their lot, each other, their families, other adults, and the socio-economic conditions of the life which they face. At the same time the students should consider methods of improving their ways of learning and living in the community of which they are a part. Cooperative democratically conducted social action projects make a valuable contribution to the articulation of the individual, the group, and the community.

c. Factors Affecting the Mathematics Program

Increase in Enrollments in Secondary Schools

In recent years there has been a tremendous increase in school enrollments, first in the elementary schools, then in junior high schools, and finally in senior high schools. Table I presents data showing changes in enrollments in secondary schools since 1890. The recent rapid growth is due partly to the extraordinary increase in the birth rate since 1940. Changes in social and economic conditions have also greatly reduced the elimination of students from the schools to secure employment.

TABLE I

ENROLLMENTS IN SECONDARY SCHOOLS—GRADES 7-12

Year	Boys	Girls	Total
1890	85,943	117,020	202,963
1910	398,525	516,536	915,061
1920	891,469	1,107,637	1,999,106
1930	2,522,816	2,689,363	5,212,179
1938	3,633,319	3,824,726	7,458,045
1946	3,248,960	3,612,070	6,861,030
1952	3,797,550	3,895,590	7,693,140

The schools face the problem of mass education in the truest sense of the word. Educational programs must be devised for the growing mass of youth that will meet their needs. The great interest of our citizens in this problem is shown by their widespread participation in the study of ways of improving the total educational program of the community.

The Increasing Length of School Life

Most young people below the ages of 17 or 18 are practically excluded from gainful employment and are required by law to attend schools. Consequently the student body at all levels includes a far wider range of academic ability and of types of talents, interests, needs, and life goals than ever before.

In most urban communities adolescents must pass through a prolonged period of actual nonparticipation in the work of the world. The school should therefore give them every opportunity to learn about social, economic, civic, and vocational problems of the community and the conditions under which they later on are likely to be confronted. Direct first-hand experience in dealing with their own problems will prepare youth to participate actively in the solution of problems of the larger community. Teaching the effective application of quantitative procedures in the defining and solution of their problems is an important contribution that the mathematics program can make to education for citizenship.

Effects of Current Promotion Policies

The lengthening of school life and the consequent increased enrollment in mathematics classes in the upper grades are also affected by existing promotion policies. In many, perhaps most, elementary schools whose primary functions may be regarded as general education, the per cent of pupils that are required to repeat the work of a grade has been reduced to a minimum. As a result children in the lower grades move from grade to grade regardless of their levels of achievement in such important

fields as arithmetic and reading. This policy of "uninterrupted continuity" has also been adopted in Grades 7 and 8 in many junior high schools. Large numbers reach the junior high school who are not prepared to do the work required in standard courses in junior high school mathematics. Special adjustments of the program are needed in such cases. As a consequence there is very little elimination because of non-promotion before Grades 9 and 10 are reached, a factor that has also contributed to the rapid increase of enrollments in junior high schools.

The problem of evaluating learning as a basis of promotion becomes complicated when faced by teachers of academic courses in ninth grade mathematics, such as algebra. These courses lie in the field of special education and are intended for those who plan to enter fields requiring special proficiency in aspects of mathematics that are beyond the requirements of the average citizen. Promotion in these academic courses depends on the aptitude and competency of the student in the particular course. When the achievement of a student is low in an academic course, the problem of "passing" or "failing" arises. A "mark" based on actual achievement suitably measured is an index of value for guidance purposes. Automatic promotion regardless of level of achievement is not regarded as an acceptable policy in areas of special education, such as algebra. However, there are other areas of general education, such as social studies, music, and the like in which the plan of continuous promotion is theoretically acceptable for the typical student.

Criticisms of the Mathematics Program

Criticisms of the mathematics program by the armed forces, business men, colleges, and others have been common and frequent. The basis of these attacks has usually been the apparent lack of mastery of mathematical skills by youth at the time they leave school. Competent research has shown that the computational skills of boys and girls who in many cases have had little if any mathematical training after completing the eighth grade deteriorate greatly due to disuse. However, experimental studies

also have shown that these skills can be quickly reconstructed under an intensive, well planned program of diagnostic teaching and review. To assure at least a fair degree of competency in arithmetic of pupils at the time they are about to leave school to take their places in their chosen occupations, or to enter institutions of higher learning, many schools now offer advanced or review arithmetic courses in which high school students may enroll during their last semester in school. In some schools levels of achievement in arithmetic are set up as prerequisites for admission to courses in bookkeeping, shop mathematics, and practical arts courses. Those requirements can be met by enrollment in systematic courses in basic mathematics in Grade 9. Steps are also being taken to maintain arithmetic skills in all basic mathematics courses and vocational courses in the high school.

The Changing Mathematics Curriculum

The nature, scope, and organization of the mathematics curriculum have been altered in many ways to meet the needs of the greatly expanded enrollments in the upper grades. The broadened conception of objectives outlined earlier in this chapter has considerably altered the contents of instructional units as is inevitable when the mathematical and social phases of the subject as well as the growth of the learner are all given adequate consideration. Special studies have been made of the needs of youth as well as of society as a basis of selecting the contents of courses, and the contents are becoming more functional and vital. Investigations of the difficulty of the various processes and types have been of value in determining the sequence and organization of the work at the various grade levels. This information has also been of value in efforts that have been made to recognize individual differences in aptitude in planning the curriculum for classes of varying levels of ability. It has led to the deferment of such topics in arithmetic as difficult operations with decimals and per cents for one or more years and the extension of the arithmetic program especially for the less able students to at least the ninth grade.

Attempts to integrate instruction in mathematics with other curriculum areas have led to such movements as the core-curriculum, the common-learnings program, and the life-adjustment program. These and related curriculum developments will be discussed in detail in Chapter 2.

Changes in the Gradation of Curriculum Content

At the elementary school level there has been a considerable shift in the past twenty years in the grade placement of many arithmetic topics. For example, instead of teaching division by two- and three-place divisors in Grade 5 as was formerly the common practice, the work in this process now is usually spread out over three years, including Grades 5 to 7. The work with easier steps in division by two-place numbers, for instance, division by even tens ($20\overline{)156}$) is begun early in Grade 5 while the most difficult steps in division by two-place divisors ($27\overline{)17{,}014}$) in which the estimated quotient must be corrected are taught in Grade 6, and division by three- and four-place divisors is delayed until Grade 7. Similarly the more difficult sequences of multiplication and division of decimals are usually delayed until Grade 7. In most schools work with per cents is usually delayed until Grades 7 and 8 where the work has more meaning for the students because of their backgrounds.

This shift of topics has added considerably to the content of what must be taught in junior high school classes. There is too much to be taught to the slower pupils in two years. Therefore many schools have adopted the plan of extending the work in arithmetic for these students through three years including Grade 9. Carpenter[3] showed that about 50 per cent of high school graduates in California had no training whatsoever in mathematics above Grade 8. In fact the maintenance of arithmetic skills has become an essential objective of all courses in mathematics in secondary schools, including "refresher" arithmetic courses for students who drop out. Extending the mathematics program in this way would make it certain that all high

[3] Carpenter, Dale "Planning a Secondary-Mathematics Curriculum to Meet the Needs of All Students," *The Mathematics Teacher*. 42:45.

school students would have at least three years of mathematics above the elementary school.

Modern Views of the Nature of Learning

Changing conceptions as to the nature of learning and the conditions under which learning best takes place are decisively affecting methods of teaching mathematics in the upper grades. There was a time when repetitive "drill" was regarded as the most effective way of developing and establishing basic mathematical concepts and skills. It also was believed that a major skill should be broken into specific elements, each to be mastered as a separate isolated item. Learning was regarded as a mechanical process, and the meaning and social values of the skills being learned were given little consideration. However, in recent years it has been demonstrated that whatever is to be learned should have its roots in some challenging need or grow out of a realistic social situation. The learner should be led to make discoveries of facts, concepts, and procedures. The elements being learned are continuously being restructured into new patterns, thus leading to insight (understanding) by the learner. Thus it is believed that what is being learned thereby is made meaningful. Learning is regarded as a growth process in which there is a change of behavior from the random confused reactions of the beginner to the controlled responses, reflecting insight, that we associate with maturity and competency. Systematic practice following understanding is then arranged to develop skill and proficiency. The contrasts between traditional and modern methods of teaching are discussed in detail in Chapter 3.

Improvements in Aids to Learning

There was a time when the textbook was the only kind of aid to learning that was provided. Often the contents were highly formal and abstract. Today forward looking teachers regard the mathematics classroom as a learning laboratory. Here the learner, under teacher guidance, discovers facts, concepts, relationships, principles, and ideas. To assist him in making

discoveries a wide variety of manipulative materials, visual aids, and learning guides such as modernized textbooks and special kinds of workbooks are provided. The value of a wide variety of community resources as a means of vitalizing instruction in mathematics and making it realistic and meaningful is widely recognized. In this way learning through direct contact with reality is closely integrated with vicarious learning through the study of materials found in books and other printed sources. In Chapter 4 the use of many valuable learning materials is discussed.

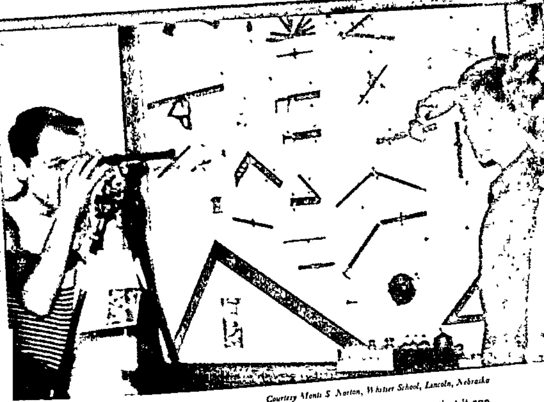
Methods of Adapting Instruction to Individual Differences

The large range in ability, level of achievement, interests, background of experience, and social status of adolescents has affected the organization of the curriculum and methods of instruction in many ways. It has been shown that the range in ability in arithmetic at the seventh grade level is between six and seven years.[4] The range increases in higher grades. The attempts to differentiate the curriculum in large junior high schools, according to the abilities of the students, particularly in Grade 9, will be discussed in Chapter 2. In small schools where the number of students is too small in a given grade to make it possible to organize several sections, differentiations of content, methods, and materials must be made by the individual teacher in the classroom in terms of the needs and aptitudes of the various students. The necessity of enriching instruction for the gifted and of facilitating learning by the slower students present many problems to teachers. Possible methods are discussed in Chapter 14.

The Improvement of Techniques of Evaluation and Diagnosis

Changes in procedures for evaluating learning in mathematics have paralleled the broadened conception of educational outcomes already discussed. Paper and pencil testing procedures that were suitable for appraising specific mathematical knowledge and skills are not appropriate for evaluating less tangible

[4] Beck, R. H., Cook, W. W., and Kearney, N. C. *The Curriculum in the Modern Elementary School*, p. 28. New York: Prentice-Hall, 1953.



Courtesy Monti S. Norton, Whittier School, Lincoln, Nebraska

Some instructional material may be displayed on a board or a rack so that it can be instantly removed and used for class or individual instructional purposes.

outcomes such as problem-solving ability, interests, attitudes, and understandings. Evaluation has become an integral part of the learning program in which both teacher and student participate. Diagnostic tests used for the purpose of identifying weaknesses so that they can be corrected promptly are regarded as essential instructional tools. In Chapter 13 we discuss important methods of appraisal and diagnosis which every mathematics teacher should know and be able to apply.

Problems in the Education of Teachers

The implications of the issues that have been discussed in the preceding pages have an important bearing on the education of mathematics teachers. A comprehensive discussion of this problem is contained in a recent report by Grossnickle.[5] He showed

[5] In *The Teaching of Arithmetic*, Chapter 11. Fiftieth Yearbook of the National Society for the Study of Education, Part II. Chicago: University of Chicago Press, 1951.

that in many teachers colleges the program that is provided is wholly inadequate for preparing prospective teachers to deal effectively with the widely varied situations they will face in their schools. They will be required to participate actively in curriculum making programs; they will be expected to plan a program of instruction which will provide for differences in the needs, abilities, and future destinies of the students; they must be able to utilize many new kinds of instructional materials; they must be skillful in the utilization of guidance in helping students to plan their future programs. They must also be psychologists and sociologists. They must be aware of the nature of the problems that are faced by adolescents and the broader community. They must appreciate the role of mathematics in social progress. All of these requirements and many others that could be listed are certain to result in marked changes in the future education of teachers.

Questions, Problems, and Topics for Discussion

1. How can a program in mathematics be adjusted to the needs, abilities, and interests of students? (This is for preliminary discussion to explore views of groups of teachers and college students.)

2. Give illustrations showing what is meant by each of the outcomes of instruction listed on page 3. Do you regard the differentiation between the mathematical and social phases as desirable?

3. What are the goals of instruction in algebra insofar as general education is concerned? What should the outcomes be for students who are planning to prepare themselves for the study of more technical work in special fields of study or occupations?

4. What is meant by problem solving?

5. Discuss the meaning of the six major functions of the junior high school listed on page 6. What is their significance as far as mathematics is concerned? Should mathematics be taught by special teachers without relation to the uses of mathematics in other curriculum areas?

6. How well does the program of mathematics in some local school take into consideration these six functions?

7. How do you account for the great increase in the enrollments in secondary schools? What is its significance for future developments in the mathematics program? What about the supply and demand for teachers of mathematics?

8. Should mathematics students ever be required to repeat a basic course? How valid is the promotion policy of "uninterrupted continuity"?

9. What changes have taken place in the mathematics program in local schools? Why? What other changes are planned?

10. How do current schemes of curriculum organization, such as those given on page 14, affect mathematics instruction? Investigate how this problem is dealt with in local schools.

11. What is meant by "learning through experience"? In what sense is learning a growth process?

12. What new kinds of learning aids are there in mathematics classrooms?

13. How does the wide range of differences among students affect the instructional program in mathematics?

14. What methods of appraising student progress and diagnosing learning difficulties in mathematics are used in our schools today? Should students receive "marks"?

15. How adequate are present day requirements for certification of teachers? Are changes necessary?

16. How effective is the local program of in-service education for mathematics teachers? How can it be improved?

Suggested Readings

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Chapter 2

The Mathematics Curriculum

In this chapter the following topics are discussed:

a. The development and status of the mathematics curriculum
b. Social and personal needs as a basis of selecting curriculum content
c. The gradation of curriculum content
d. The organization of the mathematics curriculum
e. Evaluating and improving the curriculum.

a. The Development and Status of the Mathematics Curriculum

The Early History of the Mathematics Program

In the past three centuries many changes have taken place in the fundamental philosophy and practices of our American schools. They are reflected in the mathematics curriculum. Arithmetic found its place in the curriculum in the eighteenth century due to the demands of business and industry for competency of their workers in this field. Algebra, geometry, and trigonometry later were added to the curriculum of the academy because of their practical and cultural values. Later their value as mental disciplines was stressed.

By 1860 the program of the public high school began to prevail over that of the traditional academy. In an effort to meet social demands numerous courses were set up, sometimes not well constructed or organized. The recognition of the limitations

of the existing programs of elementary and secondary schools led to a movement for reform which found its expression in 1894 in the famous report of the Committee of Ten. For the next two decades important studies were made of personal, social, and industrial needs in the field of arithmetic for the purpose of determining the minimum essentials of arithmetic. These investigations led to the elimination of numerous topics and processes that were demonstrated to be of little social value.

Prior to the advent of the junior high school, the mathematics course of study in Grades 7 and 8 was largely arithmetic, and in Grade 9, algebra. The influence of the junior high school movement has led to attempts to develop a more generalized exploratory course in these three years as the basic program in mathematics. There are five important trends in the selection and organization of the contents of the mathematics curriculum: (1) integration of the various branches of mathematics rather than the previous compartmentalization of the special subjects; (2) the consideration of the contribution of mathematics to the other curriculum areas and to life in the community; (3) the motivation of learning by integrating the consideration of the mathematical and social phases of the various fields of study so that learners can see the usefulness of the topics being studied; (4) better articulation of the work in mathematics at all levels of the school; and (5) the exploration of interests, aptitudes, and attitudes of students as a basis for more effective guidance[1] in planning the educational program of individual students.

Influence of Committee Reports

In the last few decades there have appeared several important national committee reports which have significantly affected the contents and organization of the modern mathematics program in secondary schools. The first of these was the report of the National Committee on Mathematical Requirements in 1923. This commission recommended that the mathematics curriculum for Grades 7 and 8 be regarded as a unit, with arithmetic,

[1] See *Guidance Pamphlet in Mathematics for High School Students* Washington, D C.: National Council of Teachers of Mathematics (Revised in 1953)

intuitive geometry, and graphical representation its chief components, and that in Grade 9 elementary algebra should be the major area of study. However the Commission insisted that Grade 9 should not be regarded as the terminal year in mathematical training. The role of mathematics in vocational education and other fields of study was pointed out.

In 1940 there appeared the report of the Joint Commission of the Mathematical Association of America and the National Council of Teachers of Mathematics, entitled *The Place of Mathematics in Secondary Education*.[2] The report of the commission went beyond the recommendations of the 1923 report. The specific recommendations of this report may be summarized as follows:

(1) The program should provide for continuity of instruction in many major ideas of mathematics.

(2) From the beginning mathematics should be presented as a mode of thinking about quantitative aspects of social situations and ways of dealing with them.

(3) Special emphasis should be placed on such important learning outcomes as attitudes, interests, and appreciations.

(4) Basic mathematical concepts should be introduced early and their development regarded as being gradual until understanding is attained.

(5) Provision for the integration and correlation of the mathematical subjects with each other and with other areas of the curriculum should be made.

(6) Each year emphasis should be placed on one subject, but no subject should ever be completely dropped, thus assuring the maintenance of skills and concepts previously presented and enriching mathematical experiences.

The report contains a planned outline of a course incorporating these recommendations. It also contains a grade placement chart for the contents of mathematics courses for Grades 7 to 12, with the details for each grade grouped under the following headings:

1. Arithmetic (number and computation)
2. Geometry (space perception, demonstration)

[2] Published by Bureau of Publications, Teachers College, New York, N. Y.

3. Graphic representation
4. Algebra
5. Trigonometry
6. Mathematical modes of thinking, traits, attitudes, and types of appreciation
7. History of mathematics
8. Correlated mathematical projects and activities.

An examination of the gradation chart shows that the commission made a definite effort to weave the various threads of mathematical learnings into a closely knit pattern of experiences so that instructional programs in mathematics based on it are likely to present an integrated developmental approach to the study of mathematics rather than a compartmentalized treatment of the special fields into which the contents have in the past usually been divided.

The recommendations of this commission are reflected in the contents of many textbooks and courses of study for mathematics in Grades 7, 8, and 9. Their authors have attempted with varying degrees of success to effect an integration of the various branches of mathematics through the fusion of arithmetic, algebra, intuitive geometry dealing with design and measurement, simple elements of numerical trigonometry, elementary statistics, and in a few cases sufficient demonstrative geometry to acquaint the students with the nature of formal proof. Judging from the differences in the approaches used by different authors, it is apparent that the program of mathematics in Grades 7 to 9 still is in a state of flux.

b. Social and Personal Needs as a Basis of Selecting Curriculum Content

Social Pressures Affecting the Curriculum

There have been serious criticisms of the mathematics program of our schools by parents, business, industry, the armed forces, and the faculties of high schools and colleges, as well as by students enrolled in the various courses. The results of important research showed that large numbers of students had no training

in mathematics beyond Grades 8 and 9, and that, due to disuse, the skills that they may at some time have possessed had seriously deteriorated. One study of the arithmetic ability of high school seniors led to the following conclusions:

> The results of carefully constructed tests in mathematics administered to high-school seniors and men in the Army throughout the country reveal an unsatisfactory condition. For example, on a 30-item test in the four fundamental arithmetic operations and percentage, all of the skills included being judged socially useful, the median score for over 90 high schools in all parts of the country was 17.3 items correct, or 58% correct. Obviously, this is an unsatisfactory level of performance for those about to enter any branch of military life in which a fairly high degree of computational accuracy is desirable. The level of accuracy of a large group of Army Engineers was 85%, which might be regarded as a minimum standard for our schools.[2]

This deficiency was especially evident among inductees into the war-training program. This is revealed by the following quotation from a letter by Admiral Nimitz published in *The Mathematics Teacher:*

> A carefully prepared selective examination was given to 4200 entering freshmen at 27 of the leading universities and colleges of the United States. Sixty-eight per cent of the men taking this examination were unable to pass the arithmetical reasoning test. Sixty-two per cent failed the whole test, which included also arithmetical combinations, vocabulary, and spatial relations. The majority of failures were not merely borderline, but were far below passing grade. Of the 4200 entering freshmen who wished to enter the Naval Reserve Officers' Training Corps, only 10% had already taken elementary trigonometry in the high schools from which they had graduated. Only 23% of the 4200 had taken more than one and a half years of mathematics in high school.
>
> . . . . . .
>
> The experience which the Navy has had in attempting to teach navigation in the Naval Reserve Officers' Training Corps Units and in the Naval Reserve Midshipmen Training Program (V-7) indicates that 75% of the failures in the study of navigation must be attributed to the lack of adequate knowledge of mathematics.

Since mathematics is also necessary in fire control and in many other vital branches of the naval officer's profession, it can readily be understood that a candidate for training for a commission in the Naval Reserve cannot be regarded as good material unless he has taken sufficient mathematics.

The Navy depends for its efficiency upon trained men. The men are trained at schools conducted for this purpose and the admission of men to these schools is based upon the meeting of certain carefully established requirements. However, in order to enroll the necessary number of men in the training schools, it was found necessary at one of the training stations to lower the standards in 50% of the admissions This necessity is attributed to a deficiency in the early education of the men involved. The requirements had to be lowered in the field of arithmetical attainment. Relative to the results obtained in the General Classification Test, the lowest category of achievement was in arithmetic.[4]

A special study was made to determine the minimum mathematical equipment necessary for an inductee. On the basis of conferences with 274 training officers in all branches of the armed forces, aviation, and vocational training for war-production workers, a basic list of 29 mathematical concepts was set up. This list represents the results of the first broad study of the mathematical requirements in a wide range of occupations that has as yet been undertaken. Others are needed. The implications of the list are significant, since they point the direction of possible steps that can be taken to develop a more functional program of instruction in mathematics in our schools. The committee emphasized the close parallel between wartime and peacetime needs of citizens.

Because of the value of the list of basic concepts as a guide to instruction and curriculum making, and as a basis of a self-check by individual students it is given below in the form in which it originally appeared. The list obviously stresses the social and technical aspects of mathematics; the personal needs of the individual are not given adequate consideration. They are hinted at in concepts 27 and 28. The list also appears in a special guidance bulletin prepared by the Commission on Post-War Plans

[4] Letter from Admiral Nimitz, quoted in "The Importance of Mathematics in the War Effort," an editorial in *The Mathematics Teacher*. 35.88–89.

of the National Council of Teachers of Mathematics[5] and in the Twenty-second Yearbook.[6]

1. *Computation.* Can you add, subtract, multiply, and divide effectively with whole numbers, common fractions, and decimals?
2. *Per cents.* Can you use per cents understandingly and accurately?
3. *Ratio.* Do you have a clear understanding of ratio?
4. *Estimating.* Before you perform a computation, do you estimate the result for the purpose of checking your answer?
5. *Rounding numbers.* Do you know the meaning of significant figures? Can you round numbers properly?
6. *Tables.* Can you find correct values in tables; e.g., interest and income tax?
7. *Graphs.* Can you read ordinary graphs: bar, line and circle graphs? the graph of a formula?
8. *Statistics.* Do you know the main guides that one should follow in collecting and interpreting data; can you use averages (mean, median, mode); can you draw and interpret a graph?
9. *The nature of a measurement.* Do you know the meaning of a measurement, of a standard unit, or the largest permissible error, of tolerance, and of the statement that 'a measurement is an approximation'?
10. *Use of measuring devices.* Can you use certain measuring devices, such as an ordinary ruler, other rulers (graduated to thirty-seconds, to tenths of an inch, and to millimeters), protractor, graph paper, tape, caliper micrometer, and thermometer?
11. *Square root.* Can you find the square root of a number by table, or by division?
12. *Angles.* Can you estimate, read, and construct an angle?
13. *Geometric concepts.* Do you have an understanding of point, line, angle, parallel lines, perpendicular lines, triangle (right, scalene, isosceles, and equilateral), parallelogram (including square and rectangle), trapezoid, circle, regular polygon, prism, cylinder, cone, and sphere?
14. *The 3-4-5 relation.* Can you use the Pythagorean relationship in a right triangle?

[5] Revised in 1953.

[6] *Emerging Practices in Mathematics Education*, pp. 36–38 Twenty-second Yearbook of the National Council of Teachers of Mathematics. Washington, D. C.: The Council, 1954.

15. *Constructions.* Can you with ruler and compasses construct a circle, a square, and a rectangle, transfer a line segment and an angle, bisect a line segment and an angle, copy a triangle, divide a line segment into more than two equal parts, draw a tangent to a circle, and draw a geometric figure to scale?
16. *Drawings.* Can you read and interpret reasonably well, maps, floor plans, mechanical drawings, and blueprints? Can you find the distance between two points on a map?
17. *Vectors.* Do you understand the meaning of vector, and can you find the resultant of two forces?
18. *Metric system.* Do you know how to use the most important metric units (meter, centimeter, millimeter, kilometer, gram, kilogram)?
19. *Conversion.* In measuring length, area, volume, weight, time, temperature, angle, and speed, can you shift from one commonly used standard unit to another widely used standard unit; e.g., do you know the relation between yard and foot, inch and centimeter, and similar relationships?
20. *Algebraic symbolism.* Can you use letters to represent numbers; i.e., do you understand the symbolism of algebra—do you know the meaning of exponent and coefficient?
21. *Formulas.* Do you know the meaning of a formula—can you, for example, write an arithmetic rule as a formula, and can you substitute given values in order to find the value for a required unknown?
22. *Signed numbers.* Do you understand signed numbers and can you use them?
23. *Using the axioms.* Do you understand what you are doing when you use the axioms to change the form of a formula or when you find the value of an unknown in a simple equation?
24. *Practical formulas.* Do you know from memory certain widely used formulas relating to areas, volumes, and interest, and to distance, rate, and time?
25. *Similar triangles and proportion.* Do you understand the meaning of similar triangles, and do you know how to use the fact that in similar triangles the ratios of corresponding sides are equal? Can you manage a proportion?
26. *Trigonometry.* Do you know the meaning of tangent, sine, cosine? Can you develop their meanings by means of scale drawings?
27. *First steps in business arithmetic.* Are you mathematically conditioned for satisfactory adjustment to a first job in business; e.g., have you a start in understanding the keeping of a simple account, making change, and the arithmetic that illustrates

the most common problems of communications and everyday affairs?

28. *Stretching the dollar.* Do you have a basis for dealing intelligently with the main problems of the consumer; e.g., the cost of borrowing money, insurance to secure adequate protection against the numerous hazards of life, the wise management of money, and buying with a given income so as to get good values as regards both quantity and quality?
29. *Proceeding from hypothesis to conclusion.* Can you analyze a statement in a newspaper and determine what is assumed, and whether the suggested conclusions really follow from the given facts or assumptions?

Note: The authors suggest that items 11, 14, 17, 25, and 26 are difficult to defend in a program for the less able students.

Mathematics and Consumer Education

Another report of unusual value to those interested in socializing and vitalizing the mathematics curriculum is entitled, "The Role of Mathematics in Consumer Education."[7] This report suggests that at the junior high school level such topics as the following in the field of consumer education should be taught "on an appreciation level":

1. A family's relation to the neighborhood bank
2. A personal expense account
3. A family's rationing problems (Note: out of date today)
4. The cost of operating a family car
5. The arithmetic of travel and transportation
6. Personal thrift
7. Insurance—automobile, fire, life, hospitalization
8. Growth of money by compound interest.

The following series of topics for senior high school is discussed in detail in the report:

1. Statistics
2. Consumer credit
3. Better buymanship
4. Budgets

[7] Distributed by the National Council of Teachers of Mathematics, 1201 Sixteenth St., N. W., Washington 6, D. C.

5. Insurance against numerous hazards
6. Taxation
7. Wise management of money
8. Business dealings of the home
9. Proper use of scarce or precious materials
10. The judicious use of services.

The numerous units in consumer education in present day mathematics courses and the many special courses in this field indicate a growing realization that increased consideration should be given to the mathematical aspects of problems of daily life.

In general it is recognized that the management of the affairs of individual and group living in a dynamic industrial society demands a high level of economic competence. The elements of economics to which instruction in mathematics in the upper grades can make valuable contributions are as follows:

1. Thrift and intelligent money management
2. Wise selection and use of goods and services
3. Conservation and protection of human and material resources
4. Knowledge of productive processes which help us to increase real income
5. Knowledge of the basis of common business practices and terms
6. Interest in and concern about the use of money raised by taxation
7. Support and use of the increasing variety of community and governmental services
8. Disposition to secure and utilize reliable data in dealing with socio-economic problems
9. Understanding of the fundamental relationships and interdependence of social and economic life
10. Realization of the necessity of greater economic cooperation between nations
11. Recognition of the desirability of using intelligence to give direction to socio-economic change.[8]

Studies of the Social Value of Computational Skills

Many studies have been made to determine the social value of mathematical skills. Among the most valuable of the many

[8] Brueckner, Leo J. and Grossnickle, Foster E. *Making Arithmetic Meaningful*, p. 74. Philadelphia: The John C. Winston Company, 1953.

approaches used in these investigations[9] are the following:

1. Studies of personal and occupational needs of adults
2. Need of mathematics by the armed forces
3. Need of mathematics in other curriculum areas, such as science, social studies, and health
4. Uses of mathematical skills by youth in activities in life outside the school
5. Interests of youth in mathematics
6. Mathematics found in books, newspapers, cookbooks, and other printed sources.

In general it has been found that the traditional curriculum had assimilated over the years many elements of doubtful, if any, current social value. For instance, Wilson reported the results of series of studies of the common fractions used in various occupations, showing that the fractions $\frac{1}{2}$, $\frac{1}{4}$, and $\frac{3}{4}$, accounted for 75 per cent of all fractions used. The results of a similar study by Russell[10] of the use of decimals by 68,041 workers in occupations grouped under twenty census headings are of interest in this connection.

Russell pointed out that persons employed in metal industries, trade organizations, and miscellaneous manufacturing had a much greater use for decimals than those in other occupations. In fourteen of the occupations, or 70 per cent of the list, less than 10 per cent of the population surveyed used decimals in any way; in ten of these groups the figures were 5 per cent or less. The small number computing with decimals in many of the occupations is very striking indeed.

On the other hand decimals are frequently encountered by students in textbooks[11] in the social studies and science, reference books, newspapers, and other printed sources. Understanding

[9] See bulletins, "Why Study Mathematics," and "Math at General Electric," distributed by General Electric Co., Schenectady, New York. See also the bulletin "Mathematics and You," prepared by the Mathematics Department of Rutgers University and the State University of New Jersey, 1953.

[10] Russell, G. B. "Decimal Usage in the Occupational World," *Journal of Educational Research*. 38:633–638.

[11] Hellmich, E. W *The Mathematics in Certain Elementary Social Studies in Secondary Schools and Colleges*. Teachers College Contributions to Education, No. 706. New York: Bureau of Publications, Teachers College, Columbia University, 1937.

of the meaning of decimals and the ability to read them apparently are important for certain areas of school work.

In the past forty years as a result of these and similar studies many useless topics have been eliminated from the mathematics curriculum. At the same time additions are being made to the mathematics program as new uses emerge and improved techniques are devised. It is now quite generally agreed that skill in the manipulation of fractions and decimals can be taught most effectively in connection with the technical courses at higher levels of the school in which there is a definite need for them. This belief has led to the preparation of special units on the arithmetic needed in shop courses, and so on, in connection with the work of the courses. Because incompetence in mathematics disqualifies the worker for many highly desirable occupations, these courses have become of great value.

Student Needs and Curriculum Making

There have been many reports listing in detail the mathematics which competent well-intentioned groups and individuals believe that adolescents should be taught. These reports dealt largely with the subject matter aspect of the curriculum.

In recent years an attempt has been made to move from the mastery of subject-matter approach to the study of needs of youth as a basis of planning the mathematical program. One view of need has been called the "psychobiological" concept. This interpretation of needs is that drives, tensions, and biological urges in the individual determine action. Needs are discovered by an analysis of adolescent behavior. For example, every teacher is aware of the tensions that arise when the student is unable to learn mathematics or when he is not accepted by the group as a full fledged member. Why is mathematics an area of learning which so many students report that they dread? Do teachers overlook the desires for security, affection, recognition, and new experiences that are known to be such powerful sources of purposes and motives? The diagnosis of these needs is a major task of teachers. It is their obligation to provide for the learning

not only of the basic mathematical content but also to carry on the instructional activity in such a way that the personal-social needs of the individual students are met. These ends are most likely to be achieved if the learning experiences are rich and vital to the learner and he meets with success in achieving his goals. At the same time the interaction among the members of the group should be constructive and mutually helpful.

Another concept of need is in terms of the deficiencies or lacks of the individual as seen by adults in terms of the requirements, demands, and standards of society. Needs of this kind are discovered by the analysis of society. It seems quite obvious that both approaches have value and the conflict should be reconciled.

There has been no systematic study of mathematical needs of adolescents. However, it is obvious that they arise in connection with the activities of daily life both in and out of school. The Commission on the Secondary Curriculum attempted to provide a plan of classifying needs on the basis of the aspects-of-living concept which best expressed the "idea of personal-social relationships and continuous interaction between the individual and the environment." Four categories of needs were listed as follows: (1) personal living, (2) immediate personal-social relationships, (3) social-civic relationships, and (4) economic relationships. No special effort was made to illustrate the mathematical aspects of these categories. Alberty[12] and his students devised a plan of analyzing "trends in adolescent development" in five categories: (1) health, (2) security, (3) achievement, (4) interests, and (5) outlook on life. In these areas mathematical needs arise. From the point of view of mathematics the problem presents itself of how to take these trends into consideration in planning the mathematics curriculum. A great deal of experimental work needs to be done in this connection.

How to Determine Personal-Social Needs

The needs of individuals and groups can be approached through group-study methods supplemented by interviews and

[12] Alberty, H. *Reorganizing the High School Curriculum*, Chapter 6. New York: The Macmillan Co., 1947.

case studies. The mathematics teacher should use every means and source of information available to gain an understanding of the needs of the students. Day to day contacts in the classroom and in the group life of the school are potentially the most valuable opportunities to study the needs and behavior of individuals. The information contained in a well planned system of records, reports by associates, ratings on aptitude, attitude, and special ability tests, interest inventories, and vocational plans as well as more informal contacts are all valuable to the teacher (or faculty) who wishes to adapt instruction to the needs of each individual as a unique personality. The expressed or felt needs of the students as a basis of action cannot always be depended on, since students often are completely unaware of the stresses and strains that are blocking their progress and development. The teacher faces the problem of helping the learner to identify his needs and to take steps to meet them.

The Socialized Content of Mathematics Textbooks

Studies[13] of the contents of mathematics textbooks for the upper grades published since 1850 shows that they reflect social trends in such areas as taxation and insurance. The effects of the rapid development of practices in these fields in the past century is revealed by the continuing increase in the number of concepts that are found in these textbooks which have been published since 1945. The increase has been very marked in the past few decades, indicating an increasing awareness in authors of the desirability of informing students about trends of significance to them.

The increasing awareness of the importance of consumer education is also revealed by an analysis by Bronniche[14] of the contents of mathematics textbooks For example, Table II shows the frequency with which units dealing with 35 topics in this field appeared in 20 eighth grade textbooks. This trend may become even more pronounced in the future.

[13] Unpublished master's dissertations by Sidney Heier and Archie Green, available at the University of Minnesota Library

[14] Bronniche, Richard S. Unpublished Independent Paper, p 36 University of Minnesota.

TABLE II

FREQUENCY OF CONSUMER TOPICS APPEARING IN 20 EIGHTH GRADE MATHEMATICS TEXTBOOKS

	Number of books	Per cent
I. Cost of Consumer Goods and Services		
1. Shelter	9	45
2. Food	6	30
3. Utilities	6	30
4. Owning an automobile	6	30
5. Communication	6	30
6. Shipping goods	3	15
II. Consumer Practices		
1. Instalment buying	17	85
2. Family budgets	12	60
3. Buying lumber	9	45
4. Wise buying practices	8	40
5. Sending and carrying money	8	40
6. Small loans	6	30
7. Spending the consumer dollar	6	30
8. Obtaining a proper diet	6	30
9. Transportation and communication	6	30
10. Reading meters	5	25
11. Personal budgets	4	20
III. Business Practices		
1. Margin, cost, profit, overhead, etc.	16	80
2. Selling on commission	11	55
3. Trade discount	11	55
4. Business records	7	35
IV. Banking		
1. Compound interest	19	95
2. Lending money	17	85
3. Checking accounts	16	80
4. Savings accounts	11	55
5. Postal savings	9	45
6. Mortgages	9	45
7. Building and Loan Associations	7	35
V. Investments		
1. Stocks	18	90
2. Bonds	18	90
3. Real Estate	7	35

	Number of books	Per cent
VI. Insurance		
1. Fire insurance	19	95
2. Life insurance	18	90
3. Automobile insurance	18	90
4. Hospitalization insurance	11	55
5. Purpose and origin of insurance	10	50
6. Annuities	5	25
VII. Governmental Activities		
1. Taxation	19	95
2. Governmental services	16	80
3. Budgets	11	55

Mathematical Needs Related to Reading

The incidence of mathematical terms in reading is very large. Horn reported:

> . . . of the first 1069 words in the list compiled by Thorndike and Lorge, more than one in ten are reasonably specific arithmetical, geometrical, or statistical terms And if indefinite mathematical terms are included, the proportion is about one in four. A large number of these mathematical terms appear frequently in the texts and references in the content fields.[15]

Horn reported that such technical mathematical terms as the following were rated essential by both teachers of art and of mathematics: area, balance, breadth, circle, cube, depth, dimension, distance, horizontal, length, measure, parallel, perpendicular, rectangle, square, triangle, and unit.

Mathematical concepts frequently appear in combinations in reading which increase the difficulty of dealing with them. The ability to understand such statements as the following in which mathematical terms have relations to other facts in a larger social setting is heavily conditioned by the reader's grasp of the number system and requires functional quantitative thinking:

[15] Horn, Ernest "Arithmetic in the Elementary-School Curriculum," *The Teaching of Arithmetic*, p 10 Fiftieth Yearbook of the National Society for the Study of Education, Part II. Chicago University of Chicago Press, 1951.

The difficulty of dealing with the mathematical concepts in reading is increased by the fact that they frequently appear in combination, as: '*almost two hundred years*'; '*through many centuries*'; '*millions* of *dollars*'; '*nearly two miles wide*'; '*ranges from twenty-five to one hundred twenty-five per square mile*'; '*nine hundred square miles*'; '*three* and a *half million*'; '(the river) *falls only four inches* in *a mile*'; '. . . *less* than *twenty-four times* the *annual world production*'; '*about eighty thousand tons a month*'; '*each* dot stands for *100,000* people'; '*several thousand acres*'; '*five hundred* people *on each square mile*'; '*half a mile above sea level*'; '*hundreds* of *millions* of *board feet*'; '*average* rainfall *ranges from twenty to thirty inches*'; '*Each year from sixty to seventy per cent*'; ' . . . . land *values doubled* and *doubled again*'; '*over 1,000,000 bales* of cotton *a year* which is *about one-fifth as much* as is *normally* exported by the United States'; 'It is *estimated* that the *remaining* known deposits of oil in the United States *total some 21 billion barrels*.[15]

It appears evident that teachers of mathematics should not limit the treatment of technical, quantitative, and mathematical terms to their strictly specialized meanings but also help the students to understand their interpretation in broader social situations, especially in the reading they do in any and all courses in school. The significance of this point for teachers of all curriculum areas cannot be too strongly stressed. Every teacher in a sense is a teacher of mathematics. Similarly every teacher of mathematics is a teacher of reading.

c. The Gradation of Curriculum Content

The Difficulty of Mathematical Processes and Topics

Early in the present century it was generally recognized that failure in mathematics was one of the most frequent causes of non-promotion at all levels of the school from Grade 2 through Grade 12. This led to the study of the difficulty of number facts and processes for the purpose of securing data to be used in the more effective gradation of the subject matter and the adaptation of the content to the differences in the ability and rates of learning of the students. It was found that the elements of a particular

[15] *Ibid.*, pp. 10–11

number process differed widely in learning difficulty; also that in many cases topics were being taught with disastrous results long before many students had the ability and maturity to master them, or understand their meaning. These findings have led to important shifts in the grade placement of curriculum materials, usually to the postponement of topics to later levels than were customary in traditional schools. Illustrative studies in this field and their implications will now be discussed.

RECOMMENDED GRADATION OF ARITHMETIC PROCESSES

Mental Age*	Whole Numbers	Fractions	Decimals	Per Cents
8–9	1. Addition and subtraction facts and simple processes 2. Multiplication and division facts through threes 3. Multiplication by one-place numbers 4. Related even division by one-place numbers	1. Extending uses of fractions in measurement 2. Finding part of a number	1. Reading money values 2. Addition and subtraction of dollars and cents 3. Multiplication and division of cents only	
9–10	1. Completion of all multiplication and division facts 2. Uneven division facts 3. All steps with one-place multipliers and divisors	1. Extending use and meaning of fractions 2. Easy steps in addition and subtraction of like fractions by concrete and visual means 3. Finding a part of a number	1. Computing with dollars and cents in all processes	
10–11	1. Two-place multipliers 2. Two-place divisors—apparent quotient need not be corrected 3. Zeros in quotients	1. Addition and subtraction of like fractions; also the halves, fourths, eighths family	1. Addition and subtraction through hundredths	

*Arithmetic age can be substituted in the first column

Mental Age*	Whole Numbers	Fractions	Decimals	Per Cents
11–12	1. Three and four place multipliers 2. Two-place divisors—apparent quotient must be corrected	1. Addition and subtraction of related fractions; as $\frac{1}{3}$ and $\frac{1}{6}$; also of easy unrelated types, $\frac{1}{2}$ and $\frac{1}{3}$ 2. *Multiplication* 3. Division of whole numbers and mixed numbers by fractions	1. Addition and subtraction extended to thousandths 2. Multiplication and division of decimals by whole numbers	
12–13	1. Three-place divisors	1. Addition and subtraction of types $\frac{3}{4} + \frac{5}{6}$; $4\frac{5}{6} - 3\frac{5}{6}$ 2. All other types of division examples	1. Multiplication and division of whole numbers and decimals by decimals 2. Changing fractions to decimals, and vice versa	1. Cases I and II in percentage using whole per cents
13–14	1. Extending uses of whole numbers	1. Extending uses of fractions	1. Extending uses of decimals	1. Case III of percentage 2. Fractional per cents

* *Arithmetic age can be substituted in the first column.*

The chart given above presents a systematic arrangement of arithmetic topics according to the available evidence as to their relative difficulty, beginning at about the third grade level. This information should be considered by teachers and curriculum makers in planning the instructional program. The subject matter to be taught slow-learning groups should not be the same as for groups of highly competent learners. The data in the chart suggest adjustments possible.[17]

The Relative Difficulty of Specific Types of Operations

More detailed information is available about the differences in the difficulty of specific elements within a given process. For

[17] A detailed discussion of basic data on learning difficulty of number operation is given in Brueckner and Grossnickle *Making Arithmetic Meaningful*, Chapter 3. Philadelphia: The John C Winston Co , 1953.

instance the study by Reichert[18] of the relative difficulty of processes with decimals is an excellent illustration of procedures that can be used to determine the difficulty of the specific elements of the mathematics program. Reichert prepared a series of tests containing a carefully selected sampling of all basic types of skills in the four operations with decimals. He administered this test to about 900 pupils in Grades 6 to 8 in the Twin City area. On the basis of the mental ages of the pupils based on the Otis Self Scoring Mental Ability Test, he grouped the test results and computed the per cents of correct pupil responses on composite and specific types of examples according to seven mental age levels, ranging from below 10 to over 16. Table III gives the consolidated results for the six different parts of Reichert's tests.

The boldface figures for each process are the per cents of error that were found at the mental age that would first meet a criterion of 25 per cent of error or less as a basis for grade placement.

According to the data in Table III addition of decimals is the easiest operation while changing decimals to fractions and division of decimals are the most difficult. The level of approximately

TABLE III

PER CENTS OF ERROR ON DECIMAL PROCESSES AT VARIOUS LEVELS

	Mental Ages						
Operations	below 10–11	11–0 11–11	12–0 12–11	13–0 13–11	14–0 14–11	15–0 15–11	over 16
1. Changing fractions to decimals	64.2	50.2	37.2	28.2	**22.9**	21.0	11.4
2. Changing decimals to fractions	66.9	58.5	49.3	40.8	35.6	31.6	**21.6**
3. Addition	31.2	**21.3**	19.2	15.2	13.3	13.1	10.2
4. Subtraction	34.7	30.1	**21.2**	17.5	14.2	12.4	7.5
5. Multiplication	42.1	28.1	27.6	**19.0**	15.3	14.7	10.2
6. Division	65.0	51.1	48.0	36.9	31.6	27.9	**16.7**

[18] Reichert, E. C. *The Relative Difficulty of Examples in Decimal Fractions.* Unpublished Ph D. dissertation, University of Minnesota, 1941.

25 per cent error is found at mental age 11–0 to 11–11 for addition, and at mental age over 16 for division. Notice the gradual reduction in per cents of error as mental age advances. Learning decimals is a growth process.

Reichert's original data also contain valuable information about the difficulty of specific types of examples in each of the four operations. To illustrate, the data given below show the per cents of error for five types of examples in division of decimals at the mental age of 12–0 to 12–11 years:

Type of Example	*Per Cents of Error*
1. Even division of decimal by whole number	24.6
2. Uneven division of decimal by whole number with a decimal in the quotient	52.7
3. Division of decimal by a decimal	45.5
4. Division of integer by a decimal	51.4
5. Division of a decimal by 10 or 100	64.8

Type 1 division examples were much easier than all of the other types. Types 2, 3, and 4 are of approximately the same difficulty. The level of only 25 per cent error on these three types was not reached by the pupils tested until the mental age of 16. This indicates that the present practice of teaching all types of examples in division of decimals at the level of Grade 6 is of doubtful validity, especially insofar as slow learners are concerned.

The schools should recognize the wide variation in difficulty of the different kinds of examples in this process and spread out the content over at least Grades 6 and 7 for the average student. Adjustments should be made for slow and superior students. In the case of the slow learners it might be a wise plan to teach this operation on an information basis only, not to try to reach the high level of mastery that the more gifted can achieve. It should be pointed out that higher levels of performance are almost sure to be achieved when more effective instructional procedures are used such as are described in the chapters which

follow. However, the teacher should bear in mind the fact that not all aspects of any operation are of equal difficulty.

Similarly it is known that there is a marked difference in the difficulty of the four types of examples in addition of fractions given below:

1. Addition of like fractions
2. Addition of unlike fractions within a family, as halves and fourths
3. Addition of unlike fractions whose common denominator is a product of the given denominators, as halves and thirds
4. Addition of unlike fractions whose denominators contain a common factor, as fourths and sixths.

Examples like type 1 can easily be learned by pupils with a mental age of 10, or normal Grade 5 level. Examples of type 4 are so difficult that to insure their mastery their presentation should be delayed at least two years, say the mental age of 12 or 13. A subtraction example of the type $4\frac{3}{4} - 1\frac{5}{6}$ is so difficult[19] that a mental age of 13 is required for its mastery. Examples of such difficulty and of such little social utility should not be included in the work in arithmetic for slow learners, except for purposes of information only.

Table IV gives data concerning the average difficulty of a considerable number of examples in per cent as measured by the results of a test administered to 405 seventh grade children. The data are self-explanatory. There is a wide variation in the level of difficulty of the examples given in each part of the table. Some of the less used ideas basic to the definition of per cent were more difficult than some of the more used applications of per cent. Of the three cases in per cent the easiest is Case I, finding a per cent of a number. However there is a wide variation in the per cents of correct responses in the examples within this type. Case II, finding the per cent one number is of another, is somewhat more difficult than Case I, while Case III, finding a number with a per cent of the number given, is the most difficult of the three cases.

[19] A summary of data about the relative difficulty of all arithmetic topics is included in *Child Development and the Curriculum*, Chapters 15 and 16 Thirty-eighth Yearbook of National Society for the Study of Education, Part I. Chicago: University of Chicago Press, 1939.

TABLE IV

RELATIVE DIFFICULTY OF TYPES OF EXAMPLES IN PER CENT

(Based on Results of 405 7A Pupils)

Example	Per Cent of Error	Example	Per Cent of Error
1. Expressing Decimals as Per Cents:			
(a) .05	8.4	(d) $1.33\frac{1}{3}$	59.1
(b) $.12\frac{1}{2}$	20.5	(e) .125	63.8
(c) .2	57.5	(f) 3	65.9
2. Expressing Per Cents as Decimals:			
(a) 15%	4.2	(d) 116%	25.7
(b) 6%	12.9	(e) $166\frac{2}{3}\%$	28.4
(c) 200%	25.7	(f) 118.5%	65.7
3. Expressing Fractions as Per Cents:			
(a) $\frac{1}{2}$	4.2	(d) $\frac{2}{3}$	9.4
(b) $\frac{1}{3}$	7.9	(e) $\frac{1}{6}$	14.1
(c) $\frac{3}{5}$	9.2	(f) $\frac{1}{20}$	24.4
4. Expressing Fractions as Hundredths:			
(a) $\frac{7}{25}$	30.9	(c) $\frac{1}{50}$	48.1
(b) $\frac{9}{10}$	36.0	(d) $1\frac{1}{3}$	51.1
5. Finding Per Cents of Numbers:			
(a) $12\frac{1}{2}\%$ of 80	25.4	(d) $133\frac{1}{3}\%$ of 60	47.8
(b) 6% of 80	28.2	(e) 130% of 800	49.2
(c) 28% of 72	34.1	(f) 37.5% of 720	73.6
6. Finding the Per Cent One Number is of Another:			
(a) 8 = ___% of 32	61.2	(c) 4 = ___% of 100	72.8
(b) 15 = ___% of 75	65.7	(d) 120 = ___% of 96	82.2
7. Finding a Number with a Per Cent Given:			
(a) 20 = 20% of ___	78.3	(c) 60 = 100% of ___	89.5
(b) 12 = 5% of ___	84.7	(d) 255 = 125% of ___	96.5

The Relative Difficulty of Selected Items in Algebra

Mallory made a study to determine the difficulty of a selection of items in algebra for a large number of ninth-grade pupils as shown by the per cents of correct responses. Data are given in Table V for the selected items included in the list. The items are arranged in order of difficulty. Many others are given in

Mallory's book.[20] The data should be of value in planning courses for pupils of different levels of ability in mathematics. The data enable teachers to identify easy items as well as difficult items in each area. On the basis of this information they can adjust the contents of the work to the ability of the students.

These items have been arranged in the order of their difficulty, as determined by the results of tests administered at the end of each unit of work. Following each item there is given the per cent of pupils who attempted and succeeded with the item. This per cent shows the relative difficulty of each item for the entire group of pupils that took each test. Mallory also gives data in his study as to the difficulty of each item when the pupils are grouped according to four different levels of intelligence. Those interested in the complete results should consult Mallory's original report.

The table on pages 44–45 can be used as a basis for selecting items in algebra for inclusion in a course for slow-moving pupils. In doing so it must be remembered that even a slow-moving group is rarely homogeneous and that provision must be made for individual differences. For example (assuming a class selected as this group was), if it be assumed that items should be selected which can be done successfully by 80 per cent or more of all the pupils (like items 1–2 in Test 1 following), it is evident that part of the class will need individual help on some of the more difficult items.

It is true that a part of the class will be able to succeed with more difficult items and should be expected to study them. It seems reasonable that one can include items for average classes through the 60 per cent point (items which were done successfully by 60 per cent or more of the pupils) or even lower, provided one does not expect 100 per cent mastery of all items by the entire class. On the other hand, items which are below this level of difficulty could, if deemed important, be included, provided the teacher recognized their difficulty for the slow group and provided more careful teaching and adequate practice.

[20] Mallory, V. S. *The Relative Difficulty of Certain Topics in Mathematics for Slow-Moving Ninth Grade Pupils.* Contributions to Education, No. 769. New York: Bureau of Publications, Teachers College, Columbia University, 1939.

Table V

Relative Difficulty of Selected Items in an Algebra Test

(Per cents show correct responses.)

Test 1. Making Formulas

Write in algebraic symbols, using a, b, and c for numbers:

1. The sum of two numbers. (95%)
2. George deposits c cents each week in the bank. Express in a formula the amount S he deposits in 52 weeks. (80%)

Write a formula for:

3. A number 5 times as great as b. (75%)

Write a formula for:

4. The number of inches in f feet. (64%)
5. The number of yards in f feet. (35%)

Test 2. Evaluating Formulas

If $a = 4$, $b = 2$, $c = 1$, and $d = 0$, find the value of:

1. $\frac{a}{b}$ (96%)
2. $a - c$ (91%)
3. $3a - b$ (85%)
4. a^2b (82%)
5. The value of $2x^2 - 3x + 7$ when $x = 2$ is ______ (76%)
6. If $C = \frac{5}{9}(F - 32)$, and $F = 68$, what does C equal? (65%)
7. If $w = 6$, $x = 5$, $y = 4$, and $z = 3$, what does $(z^2 - 3)(x + y - w)$ equal? (55%)
8. The value of $2x^2 - 3x + 7$ when $x = -1$ is ______ (45%)
9. The value of $\frac{3a - 2b}{a - b}$ when $a = 3$, $b = -2$ is ______ (21%)

Test 3. Signed Numbers

1. Multiply: (95%)
$$\begin{array}{r} -4 \\ \underline{+3} \end{array}$$

2. Divide: (90%)
$$\frac{-9}{-3}$$

3. Add: (84%)
$$\begin{array}{r} +5 \\ \underline{-5} \end{array}$$

4. Multiply: (76%)
$$\begin{array}{r} -5 \\ \underline{0} \end{array}$$

5. Subtract: (80%)

$$\begin{array}{r} -2 \\ \underline{-4} \end{array}$$

6. Simplify: (69%)

$$(-1)\,(-2)\,(+3)$$

7. Simplify: (64%)

$$\frac{(-1)\,(-3)\,(-4)}{-2}$$

8. Simplify: (48%)

$$\frac{(-2)^2\,(-3)^2}{36}$$

Test 4. Fundamental Operations with General Numbers

1. $(a^3)\,(a^4) =$ (89%)
2. $\dfrac{5c - 10d}{5} =$ (84%)
3. $\dfrac{4a^2b - 8ab^2}{2ab} =$ (71%)
4. Add: $\begin{array}{r} 2x^2 - 3x + 5 \\ \underline{x^2 + 3x - 4} \end{array}$ (65%)
5. Multiply: $\begin{array}{r} a^3 - 2a^2 + 6 \\ \underline{-2ab} \end{array}$ (53%)
6. Subtract: $(x - y - z) - (x + y + z)$ (40%)
7. Simplify: $2y(y - 1) + 3(4 - y^2)$ (30%)
8. $(3ab^2)^3$ (20%)

Test 5. Equations

1. $6x = 12$ (99%)
2. $x + 2.5 = 4$ (51%)
3. $x - 4 = 16$ (92%)
4. $\frac{2}{3}x = 16$ (70%)
5. $x - 2(x + 5) = 6$ (61%)
6. $cx - c^2 - 1$ (10%)

Mallory also gives information about a considerable number of geometry items that will be found of value to those interested in the teaching of intuitive geometry.

d. Organization of the Mathematics Curriculum

Current Curriculum Patterns

The curriculum pattern of the school affects the place of mathematics in the total instructional program. In some junior high schools the curriculum is organized as departmentalized subjects and little is done to bring out the interrelationships among the various fields of study, for instance, between mathematics and science, or the social studies. In other schools the

curriculum is organized into broader areas and emphasis is placed on problems and learning experiences that cut across the various related fields. Important variations in curriculum organization are being experimented with in various places, such as core programs, common learnings, life-adjustment programs, and the use of experience units—plans that are likely to influence instruction in mathematics greatly in the near and distant future.

Regardless of the curriculum pattern that emerges, it is entirely feasible to determine the basic mathematical concepts and abilities that are most frequently needed in daily life, and also the extent to which pupils of various levels of ability can master them. Research has shown that all curriculum areas, especially science and the social studies, and also the vocational courses, make a heavy demand on mathematical knowledge and skills. It also has been shown that the ability of large numbers of pupils to deal with mathematical concepts in other fields of study is at a low level. The schools must so organize instruction as to deal most effectively with mathematical aspects of all curriculum areas.

Differentiation of Courses in Mathematics in the Junior High School

In most schools the mathematics curriculum of Grades 7 and 8 is not differentiated and is still regarded as the area of general education for all. While in some large schools plans of ability grouping or homogeneous grouping of the students in these grades are used, differentiation is largely regarded as a problem to be dealt with by the classroom teacher through various plans of adapting instruction to the needs, interests, and levels of ability of the students. In some schools attempts have been made to apply the principles of the core curriculum, life-adjustment programs, and similar new-type procedures in which related courses are grouped but relatively little progress has been made along these lines.

Much greater differentiation of instruction begins at the ninth year level. On the basis of information gathered in the course of

the preceding years through tests of intelligence and achievement, prognostic tests, interviews concerning vocational plans, parents' wishes, and recommendation by teachers, the individual pupil is guided into the one of a variety of courses that appears to fit his needs.

The organization of the mathematics curriculum will depend primarily on the school's conception of the nature of learning. If the traditional subject matter conception of the curriculum prevails, the organizational pattern will follow the design of discrete subjects offered in some sequential order grade by grade. If on the other hand the experience conception is accepted, the approach to organization will differ quite radically from that of the subject-matter centered curriculum, since emphasis will be placed on the study of problem situations and topics that do not readily lend themselves to a formal subject organization. In many classes teachers are attempting to deal not only with systematic bodies of subject matter but also to vitalize and enrich their interest by the study of problems and life situations in which the subject matter functions.

A recent survey of curriculum organization in the field of mathematics in 635 secondary schools with enrollments of 300 or more students showed that at the ninth grade level, in 45 per cent of the schools there were double-track plans, in 12 per cent three-track plans, and in only 2 schools four-track plans. It appears that in the remaining schools, or about 43 per cent of the total, there were single-track plans. In 62 per cent of the schools that had a double-track program, this program was limited to the ninth grade. It usually consisted of algebra for the college-bound students and general mathematics for the others. When there was a three-track program in the ninth year it usually consisted of algebra, general mathematics, and arithmetic.[21]

The extent to which differentiation is carried in two modern high schools in Waukegan, Illinois and Rosemead, California are discussed in the following pages.

[21] *Teaching Rapid and Slow Learners in High Schools*, pages 42–43 U. S. Department of Health, Education, and Welfare, Office of Education Bulletin 1954, No 5. Washington: U. S Government Printing Office, 1954 (Price, 35 cents.)

EIGHTEEN SUGGESTED PROGRAMS IN MATHEMATICS[22]

(in Grades 9 to 12—Rosemead, California)

For Business Use

I

Basic Mathematics
Intermediate Mathematics
Secondary Mathematics
Business Arithmetic

II

Everyday Business
Intermediate Mathematics
Machine Calculation
Bookkeeping

III

Intermediate or Secondary Mathematics
Business Arithmetic
Bookkeeping
Machine Calculation

IV

Intermediate Mathematics
Secondary Mathematics
Business Arithmetic
Bookkeeping

V

Secondary Mathematics
Bookkeeping
Machine Calculation
Office Training

VI

Basic or Intermediate Mathematics
Business Arithmetic
Machine Calculation
Office Training

For Semiprofessional Engineering or For Industrial Use

VII

Basic or Intermediate Mathematics
Secondary Mathematics
Practical Geometry
Industrial Mathematics

VIII

Secondary Mathematics
Algebra 1
Practical Geometry
Algebra 2 or Trigonometry and Solid Geometry

IX

Intermediate Mathematics
Secondary Mathematics
Algebra 1
Plane Geometry

X

Basic or Intermediate Mathematics
Business Arithmetic
Secondary Mathematics
Algebra 1

XI

Secondary Mathematics
Practical Geometry
Industrial Mathematics
Advanced Shop

XII

Secondary Mathematics
Practical Geometry
Algebra 1
Trigonometry and Solid Geometry

[22] *Emerging Practices in Mathematics Education*, pp. 91–92. Twenty-second Yearbook of The National Council of Teachers of Mathematics. Washington, D. C.: The Council, 1954. (Adapted)

XIII	XIV
Intermediate or Secondary Mathematics	Secondary Mathematics
Business Arithmetic	Algebra 1
Bookkeeping	Machine Calculation
Algebra 1	Industrial Mathematics

For Preprofessional Training in Science or Engineering

XV	XVII
Secondary Mathematics	Algebra 1
Algebra 1	Plane Geometry
Plane Geometry	Algebra 2
Algebra 2	Trigonometry and Solid Geometry

XVI	XVIII
Secondary Mathematics	Algebra 1
Algebra 1	Industrial Mathematics
Plane Geometry	Plane Geometry
Trigonometry and Solid Geometry	Algebra 2

CURRICULA AND RELATED MATHEMATICS COURSES
(in Grade 9—Waukegan)

Curriculum	*Mathematics Course*
a. General Studies	Basic, in general, mathematics
b. Commercial	Business arithmetic
c. Technical preparation	Shop mathematics
d. General college preparatory	Algebra for general college preparatory
e. Scientific and professional	Algebra for science, mathematics, and engineering majors

In the case of Waukegan, special ninth grade mathematics courses adjusted to the requirements of five different curriculums are provided. In Rosemead the differentiation is even greater. Here there are provided 18 different 4-year programs in mathematics from which the student may select under proper guidance the program that is best adapted to his needs and plans for the future.

In small high schools as high a degree of differentiation as that described for Waukegan and Rosemead obviously is impossible. In such schools differentiation within the classroom is the only feasible procedure in Grades 7 and 8. In Grade 9 teachers in small high schools who are aware of the problem have developed plans of conducting what amounts to a double-track program within a single classroom. The writers know of one case in which a single teacher in a small high school conducted a triple-track plan of algebra, general mathematics, and basic arithmetic during the same class period in a single classroom.

In a very few instances schools have attempted to develop programs in which two years of work can be completed in one year, or three years of work in two years. But such plans have generally been discarded in favor of a plan stressing enrichment for the more able students. In Chapter 14 the authors give special consideration to methods of enriching the work in mathematics for gifted students.

As a result of a less comprehensive survey of the organization of mathematics programs in 92 selected cooperating schools in 35 states and the District of Columbia, Irvin reported as follows about the programs offered:

> It was found that 74 of the 92 cooperating schools offered a multiple-track mathematics program; 16 offered a double-track program at the ninth or twelfth grade level or at the ninth and twelfth grade levels; and 38 offered 'related mathematics' in connection with industrial arts, agricultural, homemaking, or pre-nursing curriculums. These courses were offered by 43 of the schools in a two-to-four-year sequence, and 33 of the 74 were offering two or three differentiated nontraditional courses.[23]

Non-promotion in Double-track Plans

An illustration of the operation of a double-track plan at the ninth grade level is revealed by the data in Table VI which shows the per cents of non-promotion in June 1954 in junior high school mathematics in the Minneapolis public schools.

[23] *Emerging Practices in Mathematics Education*, p 85. Twenty-second Yearbook of The National Council of Teachers of Mathematics. Washington, D. C.: The Council, 1954.

Table VI

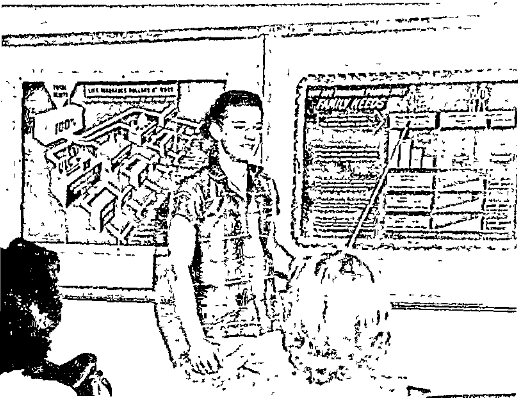
Failures in Mathematics in Grades 7, 8 and 9

Subject	On Roll	Failures	% of Failures
Mathematics—gr. 7	4,060	113	2.8
Mathematics—gr. 8	3,791	111	2.9
General Mathematics—gr. 9	1,609	44	2.7
Algebra—gr. 9	1,989	31	1.6

It is obvious that the per cents of non-promotion were very low at all grade levels. In all cases they were less than 3 per cent. They were much lower than they were twenty years ago. On the basis of a systematic guidance program ninth grade students in Minneapolis are guided into courses in either algebra or general mathematics. The result is that failures are few, especially in algebra, a course in which there has ordinarily been relatively a high per cent of failure. Here non-promotion has been reduced

Double-track programs usually contain some work in insurance.

Bancroft Junior High School San Leandro, California



almost to the vanishing point, although about 55 per cent of the ninth grade students are enrolled in the course. It should be said that there was some variation in the per cents of failure in each of the four courses among the 14 schools, ranging in Grade 7 from 0.0 per cent to 10.4 per cent, in Grade 8 from 0.0 per cent to 11.0 per cent, in general mathematics from 0.0 per cent to 13.9 per cent, and in algebra from 0.0 per cent to 3.5 per cent.

Guidance in Mathematics Classes

The problem of guiding students into the various mathematics courses presents many problems. The majority of plans reported to Irvin included a consideration of the following items as a basis of placement: (1) the student's past achievements, (2) the recommendations of teachers, (3) reading comprehension test scores, (4) special interests and vocational preferences, (5) mental test ratings, and (6) parental wishes. The eighth grade mathematics teacher is the key person in the guidance of young students.

An interesting discussion of the ways in which guidance can be carried on in the mathematics program of the junior high school is the following adaptation of a statement by Alice Hach:

1. Use the first few days of the seventh grade to acquaint children with the true nature and purposes of the junior high school mathematics program.
2. Use the grades at the end of the first grading period to point out how grades are a measure of progress and to stress the relationship of this progress to the child's future choices in mathematics. Each subsequent grading period then offers an opportunity for a child to evaluate his growth.
3. Use a check list of work habits on report cards, *only* when the habits listed can be objectively measured by the teacher and pupil. Only if this is true can an evaluation of work habits be of any value in determining the growth and development of those characteristics needed for specific types of study.
4. Use standardized tests to provide specific information for the teacher, the child, and the parents. Ways can then be provided for a child to overcome specific deficiencies It is important that the child and parents understand the place of standardized tests in the child's total mathematics program.

5. Have parent conferences early enough in junior high school to give parents an opportunity to understand the needs of a child while he is still on the single-track program in mathematics. The more nearly a parent understands the progress of the child throughout the junior high school the better able he is to help the child in making wise decisions regarding the selection of future courses.
6. Use professional articles, guidance bulletins, and all such helps as special project material. This might be offered throughout a child's mathematics program.
7. Show the relationship of the material that is being studied to future courses in mathematics. For example the section on banking, budgets, and business accounts might be related to general mathematics. Plane figures might be related to plane geometry, solids to solid geometry, while formulas might be shown as developing fundamental principles of algebra In each case the respective future textbook might be examined. This gives the child an opportunity to see not only the relationship of the new material to that which is familiar, but also to recognize its place in future courses.
8. Have students give talks regarding mathematics courses. A student from an algebra class and one from a general mathematics class might be invited to talk to the eighth grade students, and a student from a geometry class might talk to the algebra students.
9. Have a display of advanced texts with notations explaining the prerequisites needed for each of the courses and the grade level at which the course can be studied.
10. Use aptitude tests to give the pupil another measure of his chances for success in a particular course. Pupils and parents need to understand the limitations of such a test as well as its value to the pupil. It is helpful for the parents and the child to know that when aptitude tests are used with *other* measures, they can be useful devices for guidance.[24]

e. Evaluating and Improving the Mathematics Program

A Plan for Evaluating the Mathematics Program

An excellent procedure that can be used by the faculty of any secondary school to evaluate the mathematics program is the

[24] *Emerging Practices in Mathematics Education*, pp 83–84 Twenty-second Yearbook of The National Council of Teachers of Mathematics. Washington, D. C.: The Council, 1954.

application of the carefully developed series of criteria given in the volume, *Evaluative Criteria*, 1950 edition.[25] On the basis of the resulting information, the faculty can take steps to improve the instructional program. This procedure has been carried out by many faculties.

Improving the Curriculum

The process of curriculum development is under way at all times in forward looking school systems. It takes place whenever the teacher in the classroom studies the needs of the students and attempts to adjust the instructional program to these needs.

The process of curriculum change is stimulated by modern curriculum guides prepared by members of the instructional staff under the direction of curriculum departments in state departments and local school districts. The development and study of these guides is often an integral part of the program of in-service education. Study groups and special committees participate in the process of curriculum development. Conferences, institutes, and workshops attended by lay citizens and members of the staffs of the schools are frequently arranged so that there may be widespread participation in the study and evaluation of the existing curriculum and in the formulation of policies to guide future action. In many centers experimental work is under way dealing with the trial of various ways of improving the curriculum and with the actual evaluation of instructional materials and units of work that are thought to be in line with the principles of modern education. The publication of curriculum materials whose value has been demonstrated provides valuable guidance for teachers in the selection of units for their own classes. Ideally there should be more units available than any teacher can possibly use with a class in the course of a year, so that units will be selected that are most adapted to the needs of particular classes. Resourceful teachers will of course constantly be experimenting with new units of work that are adapted to the emerging needs of their students.

[25] Distributed by Cooperative Study of Secondary School Standards, Washington, D. C., 1950

In the 1944 report of the Commission on Post-War Plans[26] the following list of tentative proposals were given as guides in improving the mathematics program of our schools:

I. The school should insure mathematical literacy to all who can probably achieve it.

II. We should differentiate on the basis of needs, without stigmatizing any group, and we should provide new and better courses for a high fraction of the school's population whose mathematical needs are not well met by the traditional sequential courses.

III. We need a completely new approach to the problem of the so-called slow-learning student.

IV. The teaching of arithmetic can be and should be improved.

V. The sequential courses should be improved.

The Commission said that "provision for growth in the mastery of arithmetic should be continuous throughout the elementary and secondary schools."

Principles of Curriculum Development

The basic principles underlying the mathematics curriculum which have been discussed in the preceding pages may be summarized as follows:

1. The learning of arithmetic and other branches of mathematics is a gradual growth process that should be guided and directed at all stages by a systematic, planned program. It should begin early in the primary grades.

2. The mathematics program should include a well integrated treatment of the mathematical and social phases of the subject, dealing with topics and processes of undoubted social value and significance to the average individual. The more difficult computations such as are required in technical work should be deferred to levels beyond the elementary school.

3. The content of the curriculum should be based on personal and social needs emerging in current living both in and out of school. The evidence is clear that students at the junior high

[26] "The First Report of the Commission on Post-War Plans," *The Mathematics Teacher*. 37:226–232.

school level have many quantitative experiences that should be made mathematically meaningful and socially significant to them.

4. The student should be made intelligent about the development, status, and likely future trend of important social institutions through which number functions in the community.

5. Mathematics should be taught in close association with any and all school work in which the use of quantitative procedures will clarify the situation and help to make it meaningful.

6. A most fruitful approach to the enrichment of mathematics instruction is the consideration of significant problems that will illuminate the present social situation for the learner, particularly in the area of economic competence.

7. Growth in the ability to apply mathematical procedures effectively in social affairs is greatly facilitated by abundant experience in using number in a variety of purposeful activities.

8. Even though much arithmetic is learned incidentally through contact with number in social experiences, such learning is neither systematic nor comprehensive. It is clear that direct instruction is necessary for mastery of the basic skills and efficient work methods.

9. Systematic provision should be made for adapting curriculum content and instructional procedures to differences in the interests, abilities, and needs of the pupils, as well as differences in the rates at which they learn.

10. The curriculum should be so arranged as to provide for continuity of pupil development, with a minimum of strain and tension, and it should be so organized that there is a reasonable likelihood of successful learning. The available evidence as to the learning difficulty of number processes should be carefully considered in the gradation of subject matter.

Questions, Problems, and Topics for Discussion

1. How did the advent of the junior high school affect the mathematics curriculum?

2. Discuss the five trends in the mathematics curriculum that are given on page 21. Apply them to the local situation.

3. Describe methods of guidance useful in mathematics instruction.

4. Evaluate important committee recommendations about the mathematics program.

5. Examine textbooks in junior high school mathematics and be ready to discuss plans followed in organizing their contents.

6. What criticisms are being made of the present mathematics curriculum? By whom? How valid are they? How can they be answered?

7. Examine the list of 29 mathematical concepts given on pages 26 to 28. Does the list supply an adequate basis for setting up a program of mathematics for the average citizen? What would you omit? What would you add?

8. What is the role of mathematics in consumer education?

9. Look up and evaluate Russell's study on the occupational use of decimals.

10. How can the teacher determine pupil needs of mathematics? How about social needs?

11. Of what value are the data on gradation such as those given on pages 37 to 38? How can data in Tables III, IV, and V be used in differentiating the mathematics program in terms of the differences in ability of students? Construct a suitable test of types of examples in division of decimals or per cents and compare results with data given in this chapter.

12. How is the mathematics program in local secondary schools organized? What changes are desirable? Compare with the programs of Waukegan and Rosemead given on pages 48–49.

13. Find out about rates of non-promotion in mathematics classes in local schools. Why has the present policy of "non-interrupted continuity" been adapted in so many schools? What are some of the consequences insofar as curriculum and instruction are considered?

14. How does the program of guidance in local schools deal with the organization of mathematics classes? What is being done for the superior student? Should students be grouped according to ability?

15. How can a faculty systematically evaluate and improve the mathematics program? Review *Evaluative Criteria* referred to on page 54.

16. Discuss the principles of curriculum making given at the close of the chapter. Would you restate any of them? Would you add others?

17. Why should the mathematics program be regarded as continuous at all levels from the primary grades through the high school and beyond?

18. What can junior high schools do to create a deeper interest in mathematics on the part of its students? Why is this necessary?

Suggested Readings

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Faunce, R. C., and Bossing, N. L. *Developing the Core Curriculum*. New York: Prentice-Hall, Inc., 1951.

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Chapter 3

Principles of Teaching Mathematics

A survey of modern instructional practices in mathematics reveals several important trends: (1) a broader conception of the goals and outcomes of instruction is being adopted, (2) the point of view is being recognized that what the student learns should be mathematically meaningful and socially significant to him so that he will understand and appreciate the value of what he is learning, (3) increasingly it is being realized that learnings should be organized as larger units for study which deal with the uses of mathematics in daily life, (4) efforts are being made to develop methods of adjusting curriculum and methods of instruction to individual differences in the ability, needs, and interests of the students.

In this chapter the following topics are discussed:

a. The relation of goals to instructional procedures
b. Principles of learning applied to mathematics
c. The role of mathematics in problem solving
d. Theories of learning applied to mathematics
e. Provision for individual differences.

a. The Relation of Goals to Instructional Procedures

Types of Learnings in Educative Experiences

It is generally recognized today that there are many simultaneous learnings in any genuinely educative teaching-learning

situation. The systematic consideration by the teacher of the three kinds of learnings listed below and the selection of ways of bringing them about has a direct bearing on the nature of the instructional program in mathematics:

1. Learnings directly related to the technical aspects of mathematics
   a. Understanding of numbers of all kinds
   b. Meaning and understanding of mathematical operations and proficiency in their performance
   c. Ability to solve problems, to read and interpret graphic and tabular materials, etc.
   d. Resourcefulness in applying mathematical procedures
   e. Knowledge of basic information about social institutions and practices in which mathematical procedures function directly
2. Learnings directly related to the personal development of the student
   a. Range of interests and appreciations in the field of mathematics
   b. Attitudes and systems of values
   c. Social adjustment and behavior in group situations
   d. Mental health, emotional stability, and physical well being
   e. Skill in oral and written expression, implying clear thinking
3. Learnings of a broader societal nature
   a. Problem solving in the broader sense
   b. Sensitivity to the values and deficiencies of social processes and readiness to participate in their improvement
   c. Skill in democratic coöperation
   d. Qualities of leadership
   e. Creativeness in action.

Instructional Implications of This Analysis of Learnings

The implications for instruction of this analysis of learnings are quite [illegible] The instructional program should be so conducted t [illegible] learnings in mathematics

as such be taught, but consideration will also be given to the development of all aspects of the learner's personality, as well as the learnings of a societal nature. The latter have deep significance for effective participation in our democratic way of life. In such a program it is necessary to give special consideration to a study of the needs and interests of the students and the demands of life in the community so that a body of content can be selected for study that is certain to appeal to the learners as vital, realistic, and worthwhile. Its consideration will lead to the development of good attitudes and appreciation. Adjustments of the instructional content and procedures must be made to the ages, maturity, and learning ability of the students so that they are most likely to be successful in their study of mathematics. Emphasis will be placed on the teaching of meanings so that there will be insight and understanding.

To assure the achievement of the outcomes of a broad societal significance listed above, learning experiences must be provided in which the students as members of a group study topics related to the social institutions through which mathematics functions in the affairs of life. Mathematics is often a determining factor in the consideration of ways of dealing effectively with problems in their own lives, in the life of the school, and in the larger community. As will be shown shortly, mathematical procedures play an important role in problem solving.

It is obvious that a narrow instructional program which neglects meanings and emphasizes the learning of number facts and mathematical processes through rote memorization and intensive repetitive drill procedures, with little if any consideration of their social applications, practically disregards the outcomes related to personality development and those of a societal nature.

b. Principles of Learning Applied to Mathematics

The Nature of Learning

The evidence that learning has taken place is to be found in changes in the behavior of the learner as a result of experience.

In our schools learning by students is usually guided toward specific goals through organized patterns of experience. The purpose of instruction in mathematics should be the provision of meaningful experiences and the arrangement of learning situations that will enable the learner to reconstruct his behavior in the direction of socially desirable goals that he sets up with the help of his teacher and classmates and accepts as his own. In the modern classroom the learner is called on to make discoveries, abstractions, and generalizations, and is given guidance in the organization and structuring of his mathematical skills, ideas, and concepts.

The most effective basis of learning is problem solving. Before learning can take place there must be a situation in which the student feels a need which he desires to satisfy or a difficulty which he is motivated to overcome. If he merely calls on previous learnings to meet the situation, no new learning takes place. If, however, the learner's efforts are blocked and he is not able to achieve his goal, he will consider possible ways of solving his problem, and try one or more plans of action, until finally he gives up or arrives at a solution. The more mature he is in his thinking, the more intelligent and forward looking will be his actions. At the same time, he will make less use of trial and error methods.

When the learner has solved his problem, he is ready to readjust his total behavior in terms of what he has learned. Normally with the help of his teacher and associates he goes over his solution and formulates it clearly; he examines his procedure critically; he draws conclusions and makes generalizations about his experience. Thus he develops a new pattern of behavior that will function in the solution of new problems that he may encounter in situations of a similar nature.

Problems That the Learner Must Solve

The learner faces many kinds of problems in the study of mathematics. The teacher should make every effort to introduce new topics through problematic situations that are within the experience of the students. Through these problematic situations

the students should become aware of a need or difficulty and see reasons for attempting to meet it.

These needs may be closely related either to the mathematical phase of instruction, or to the study of some topic or problem directly related to the social phase of the subject. For instance, the teacher may help students to determine weak spots in number operations requiring attention at the beginning of the year through the administration of diagnostic tests which locate existing deficiencies. Problem solving is involved in the steps that might be taken by a class to discover the relationship between the radius of a circle and the circumference and then to derive a formula. Similarly the teacher may consider with the class ways of solving some such realistic problem related closely to student interests and needs, as, "How much does it cost to operate an automobile for a year?", or "What plan can we work out at school that will enable us to get a loan of money for our lunches when we forget to bring the money from home?" In one school the solution worked out for this last problem resulted in the establishment of a credit union, conducted by students under the auspices of the student council. This student credit union was run much as credit unions are operated in life outside the school. (For a detailed discussion of how to teach problem solving, see Chapter 9.)

Needs as a Basis of Planning Instruction

Psychologically speaking a need is a state of disturbed equilibrium. When the individual himself feels keenly about the condition, he ordinarily attempts to do something to restore balance. Sometimes he may act on impulses that are superficial, capricious, and fleeting. When the individual on the other hand consciously selects a goal to be achieved in relation to the need he senses, human purpose becomes operative.

In a truly educative experience a purpose is selected by an individual, or a group, with knowledge of the possibility of its achievement, with an awareness of the requirements for its fulfillment, and knowledge of the possible consequence of failure. A desirable line of action is then selected from among those

possible. Planning the steps to be taken, carrying them out, evaluating the consequences, changing plans in the course of action as may be necessary—all these continue until the goal is attained. The purpose which the learners have selected becomes the motive or drive for continued action. Ideally all educational activities, whether they be assignments, enterprises, projects, units of work, practice exercises, or what not, should be so arranged and conducted that they suggest or initiate purposes which lead the learners to make an aggressive effort to achieve worthwhile goals.

The Social Significance of Problem Solving

When a group of learners faces a situation in which all are conscious of some common lack or difficulty, what have been variously defined as purposes, wants, desires, wishes, goals, needs, and interests emerge which impel the group to action. First the group helps the individual to locate his own need and to define it. Problem solving is a process of deliberative action in which the individual works cooperatively with others to meet and solve common needs.[1] Reflective thinking about the past helps the learner and the group to understand the present situation and provides the basis for taking more intelligent action in the future than in the past.

When people solve their problems intelligently by a mutually cooperative group process, genuinely socialized living emerges. A high level of learning takes place when in response to needs the learners intelligently and deliberately select what they learn and the behavior patterns they wish to acquire and cooperatively plan the steps to be taken to achieve their goals. Under such circumstances needs are raised to the level of conscious action directed to self improvement. The school must provide a rich stimulating environment in which students become aware of needs and their implications. Then under competent guidance they can select and plan experiences that will help them to meet these needs.

[1] The point of view here presented draws heavily on the discussion in Hopkins, L. T. *The Emerging Self.* New York: Harper, 1954.

Hopkins summarizes the evidence[2] as to the value of democratic group processes when applied to the solution of a significant problem, as against "authoritarian" procedures directed by the teacher as the one in charge of a group as follows:

1. "A cooperative group releases more potential ability than an authoritarian group."
2. "The emerging thinking has a quality superior to that of any member, even to that of the leader or the status control" that exists in a group of individuals not integrated organically about a common purpose.
3. "Since the process fosters self-control in each member, such a group has internal self discipline." The internal unity of the group and the self control of the individual complement each other, and are "so interrelated that one cannot be achieved satisfactorily without the other."
4. "The organic group has its unity in the high social-moral quality derived from acting on thinking or deliberative social action in which common consent and acceptance, active and uncoerced, is the test of behavior."
5. The group as a whole helps each individual member to clarify his concept of his own needs, refine his own meanings, to improve the logic of his experiences, and to evaluate his actions, contributions, and value judgments.

The conditions that create a group and cause it to move forward are processes which make the following possible:

1. The awareness and acceptance of a problem that is vital to the group
2. Interaction among the members of the group leading to involvement
3. Planning of activities and steps to be undertaken
4. Collective thinking involving the making of suggestions, proposing of procedures, pooling and analyzing findings, interpreting findings

[2] Adapted from the discussion in Hopkins, *op. cit.*, pp. 200–202.

5. Free discussions of ideas, beliefs, problems, values, activities, conflicts, prejudices, and attitudes leading to desirable changes and integration of behavior

6. Sharing work to be done, delegating and accepting responsibilities, willing participation

7. Keeping necessary records of findings, ideas, activities, sources of data

8. Reporting progress, findings, difficulties, actions

9. Evaluating contributions, actions, processes

10. Deciding on next steps in the light of interpretations of findings.

The significance of the group process approach to the teaching of mathematics is discussed in detail in Chapter 8. At the same time the teacher should encourage able students to explore topics that are of interest to them even though they may have no immediate social significance. Such a student may develop a high degree of insight in very abstract areas of mathematics that require very superior mathematical competency. (See Chapter 14 for a special discussion of this problem.)

c. The Role of Mathematics in Problem Solving

Basic Concepts in Problem Solving

In the important report "Mathematics in General Education"[1] it is maintained that the mathematics curriculum should be based on the identification and study of concrete problems which arise in connection with meeting needs in the basic aspects of living. Mathematical procedures play an important role in the systematic study of quantitative aspects of these situations, including the use of computational skills. The report lists seven major concepts that are basic to problem solving and discusses these concepts with special reference to their mathematical implications: (1) formulation and solution of a problem, (2) data, (3) approximation, (4) function, (5) operation, (6) proof, (7) symbolism. The following discussion is a brief summary of Part

[1] Published by Appleton-Century Co., New York, 1940.

III of this report, which the reader should consult for amplification of details as to the meaning of each concept and for a comprehensive discussion of instructional procedures that are useful in developing each concept.

1. *Formulation and solution.* The first step in the problem solving procedure is the identification of a difficulty and the formulation of the problem or of the goal sought in clear concise meaningful terms. Then a plan of action, the solution, for resolving the difficulty can be laid out. This procedure requires a high level of thinking by the group and individuals. The way the problem is formulated affects both the final solution and also the steps in the process by which it is secured. The solution may be, for example, a statement of procedure, the recall and application of a rule or formula, a drawing, a construction, or an algorism to be used.

2. *Data.* The next step is the collection of the data necessary for arriving at a solution. The data should be representative, relevant, accurate, and reliable. It is necessary to collect, record, and organize the data in ways that are appropriate to the solution of the problem. The data must then be analyzed and interpreted before it is possible to draw conclusions and generalizations from them.

3. *Approximation.* All measurements involved in problem solving are approximate. The fundamental notion of approximation is indicated by such terms as precision of measurement, accuracy, rounding off, and significant digits. A clear recognition of the approximate nature of measurements not only helps the student to exercise any necessary precautions in recording and reporting data, but also makes it possible to compute with approximate numbers with considerable economy of time and labor in making estimations or checking solutions.

4. *Function.* The notion of some sort of correspondence or relationship between two or more sets of data underlies the entire process of solution. In analyzing and interpreting sets of data, the investigator seeks to find relationships among the variables such that knowledge of the values of one or more of them serves to determine uniquely the corresponding values of some other variable. Sometimes this leads to the development of a formula

which summarizes the relationships in symbolic form. The student who discovers through experimentation the relationship between the length of the radius of a circle and its circumference is dealing with the function concept.

5. *Operation.* Problems cannot be solved without some kind of operation—computation, experimentation, or mental procedure. Mathematical operations are active processes involving the manipulation of mathematical symbols that are used to represent data. Finding the square root of a number, estimating, making approximations, and checking computations are all illustrations of operations. The choice of operation or method is guided by the concepts of approximation and function.

6. *Proof.* Insight into the relation of conclusions to initial assumptions and to defined and undefined terms makes it possible to work through a solution upon which one can rely with assurance. On the same basis one can accept or reject with confidence the solutions proposed by others.

7. *Symbolism.* Words, signs, marks and other symbols that can be used to represent concepts make many forms of reflective thinking possible. Symbols are essential for the communication of ideas to others. Symbols also facilitate the manipulation of ideas. The development of a meaningful vocabulary and the ability to use symbols intelligently should be goals to be borne in mind by all mathematics teachers.

The teacher of mathematics should guide the learning activities of the students in such a way that each of these broad basic concepts is carefully and systematically developed through actual experience in problem solving. On the basis of observation of the work of the students in problem solving situations related to any of the concepts, the teacher can discover their strengths and weaknesses in utilizing mathematical procedures and then take steps to bring about an improvement by arranging suitable learning situations. For some concepts, for example, operations, the teacher can make excellent use of standard and informal tests to determine strengths and deficiencies in mathematical operations. The important point is that all of these concepts can be developed through experiences in which they function. None emerge as if by magic.

The learning of new steps in number operations can be approached on a problem solving basis. First the learner should be brought face to face with a variety of social situations in which the need for the new operation arises. He may suggest the situations himself or they may be presented by the teacher in various ways or by the textbook. The new difficulty presented in the step should be clearly defined so that the learner is aware of the goal to be achieved and accepts the desirability of its attainment. The learner's established patterns of behavior or habitual responses are not in themselves sufficient to solve the problem.

Possible procedures of arriving at a solution or answer are explored and tried informally so as to make the task more meaningful to the student. Then under the guidance of the teacher the algorisms to be learned are developed with the class in such a way that the teacher feels assured that the students understand the sequence of steps taken to complete the solution. Insight will be demonstrated by the ability of the students to answer questions designed to test their understanding and their ability to apply the step in the solution of new problems.

An illustration of the problem solving approach to the learning of a mathematical procedure is the approach used by a teacher to the study of the subject of per cent. The teacher began the discussion of the topic by asking the students to give examples of the uses of per cents with which they were familiar. A number of specific illustrations were given. However, most of the students felt that there were other uses of per cents with which they were not familiar. So it was decided to explore systematically the ways in which per cents were used in the home, in the factory, and in business offices.

One interested group volunteered to look for illustrations of per cent in newspapers and other printed sources. On the following day reports first were made by the individuals in the class and uses were listed on the blackboard. Then the special committee made its report which was full of interesting information. As a result of their research the students became fully

aware of many of the uses of per cent in daily life. Then the teacher suggested that they examine the contents of the chapters in the textbook in use to discover what they included about the topic and the phases of the subject to be covered by their study of the topic. Many of the uses of per cents that had been listed by the class were found in the textbook. Some omissions of topics were noted which the students felt should be added to those included in the textbook. Then the systematic study of the subject began.

Types of Functional Quantitative Thinking

The highest level of *functional quantitative thinking* is that done in connection with the solving of problems of daily life that are realistic and vital to the learner, and which are clarified by mathematical techniques. However there are many times when the individual also must think quantitatively in other than problematic situations, for instance, in reading newspapers, reference books, and other printed sources; in interpreting tables, graphs, diagrams, and other ways of presenting quantitative materials in a systematic manner; and in organizing numerical data that he has gathered for some purpose. This type of quantitative thinking is of a much higher level of value than that done in reading and solving perfunctory routine types of "verbal problems" such as were formerly found in large numbers in most textbooks and standard tests.

Functional quantitative thinking is also done in other meaningful situations, such as when the student reads explanations of mathematical procedures in textbooks, discovers relationships among sets of data, number processes, and arithmetical concepts, makes estimates and approximations to check his solutions, and derives rules and formulas of various kinds.

When the work in mathematics is largely limited to repetitive drill on mathematical processes, or practice in solving routine verbal problems, functional quantitative thinking, if any, is certain to be at a low level. However, when the student is stimulated and challenged, he can have deep insight into number relationships and he can do much functional quantitative thinking.

d. Theories of Learning Applied to Mathematics

Changing Concepts in Theories of Learning

There was a time when the mind was regarded as a "reservoir" into which facts and knowledge could be poured and stored for future use. Memoriter methods were stressed under this concept, leading to the memorization of rules, tables, proofs, and the like, often without understanding by the learners or any notion on their part as to the usefulness of what was learned.

According to another theory the mind was "regarded as a muscle." The important thing was to develop this muscle through a variety of formal set exercises, again without regard to the meaning and significance of what was learned. The more difficult the exercise, the greater the mental development was believed to be.

At the present time there are two main psychological theories of how learning takes place, namely, "association" theories on the one hand and "field" theories on the other. Both theories have affected methods of organizing and teaching mathematics courses. Of the association theories Thorndike's "connectionism" or "stimulus-response" psychology has played the principal role. According to this theory learning consists in establishing "bonds" through the application of methods justified by Thorndike's formulation of the laws of learning, including exercise—use and disuse—and effect. Since it was believed by connectionists that each bond existed separately and independently, the conclusion was drawn that the various facts should be learned as isolated elements through repetitive drill rather than through the study of a systematic arrangement of facts based on their relationships. The same was true of number operations.

Sponsors of "field" theories oppose the "atomistic" approach of connectionists. Instead they emphasize understanding of the number system and its uses in number operations and problem solving in a variety of situations rather than learning through intensive drill. They also point out the importance of the organization of learnings through the discovery of relationships and generalizations among facts and processes rather than through

the study of isolated elements set up in unrelated form. According to the field theory the pupil should not practice a skill to develop proficiency until he knows the meaning of the process and understands how it operates.

The conflict between these two main theories of learning arithmetic grows out of the fact that each of them offers what seems to be a more logical explanation of how certain types of learning takes place than the other. For example, problem solving procedures seem to be more in line with field theories, while those who strongly stress drill procedures in teaching operational skills find support for their actions in connectionist theories.

There is no need of trying to force all learning situations to conform to a single theory. This is especially true when one considers the wide variety of outcomes there are of mathematics instruction. The procedures used to develop in an individual student a knowledge of basic number facts are quite different in nature from those used to develop the ability of a group of students to solve a problem growing out of a social situation which is of concern to the whole group. Similarly the instructional procedures that are used to develop the critical thinking necessary to evaluate some practice or a public policy differ greatly from those used to develop the methods of thinking used in gathering information from various sources about some point of conflict, organizing this information, and then setting up standards to guide future actions.

Meanings in Mathematics

The learning of arithmetic and other branches of mathematics is greatly facilitated when the work is made meaningful for the student and he understands the procedures he is studying. A distinction is sometimes made between the terms, "meaning" and "understanding." For instance, the student may know the meaning of the example, 7.5 + 9.8, namely, that he is to find the sum of the two numbers, but he may not understand the process of adding the two numbers, especially the step of transforming thirteen, the total of the tenths, to 1 one and 3 tenths.

Frequently the words "insight" and "understanding" are used synonymously. However, insight has an emotional tone suggested by the "ah-ha, I get it!" expression.

There are various kinds of meanings that are associated with the mathematical and social phases of arithmetic. Brownell discusses four categories of meanings related to the mathematical phase of arithmetic in the following statement:[4]

1. One group consists of a large list of basic concepts. Here, for example, are the meanings of whole numbers, of common fractions, of decimal fractions, of per cent, and most persons would say, of ratio and proportion. Here belong, also, the denominate numbers, on which there is only slight disagreement concerning the particular units to be taught. Here, too, are the technical terms of arithmetic—addend, divisor, common denominator, and the like—and, again, there is some difference of opinion as to which terms are essential and which are not.
2. A second group of arithmetical meanings includes understanding of the fundamental operations. Children must know when to add, when to subtract, when to multiply, and when to divide. They must possess this knowledge, and they must also know what happens to the numbers used when a given operation is employed. If the newer textbooks afford trustworthy evidence on the point, the trend toward the teaching of the functions of the basic operations is well established. Few changes in the more recent textbooks, as compared with years ago, are more impressive.
3. A third group of meanings is composed of the more important principles, relationships, and generalizations of arithmetic, of which the following are typical. When 0 is added to a number, the value of that number is unchanged. The product of two abstract factors remains the same regardless of which factor is used as multiplier. The numerator and denominator of a fraction may be divided by the same number without changing the value of the fraction.
4. A fourth group of meanings relates to the understanding of our decimal number system and its use in rationalizing our computational procedures and algorisms There appears to be a growing tendency to devote more attention to the meanings of large numbers in terms of the place values of their digits. Likewise there is a strong tendency to rationalize the simpler

[4] Brownell, W. A. "The Place of Meaning in the Teaching of Arithmetic," *Elementary School Journal*, 47:257–258, Chicago: University of Chicago Press

computational operations, such as 'carrying' in addition and 'borrowing' in subtraction; but there is some hesitation about extending rationalizations very far into multiplication and division with whole numbers and fractions.

A similar series of categories of meanings related to the social phase of arithmetic should be added to the above list, such as:

1. Basic concepts in the field of measurement, including historical aspects of their status and development
2. Economic competence
3. Understanding of social institutions, such as banking, taxation, insurance, etc., through which number functions
4. How quantitative procedures have assisted man in his struggle to control nature
5. Concepts necessary to the intelligent interpretation of tabular and graphic materials.

How Meanings Develop

Meanings are basically the outgrowth of experience. Young children learn the meaning of number through direct concrete experience, often involving the manipulation of materials of various kinds. Later they learn to use words, for instance, four, and later figures, 4, to express in symbolic form the meaning they wish to convey. Finally they make a generalization which enables them to apply the concept 4 in any situation.

The steps in developing concepts have been listed by Van Engen[5] as: sense perception—abstraction—generalization. He points out that as children mature and learnings become more complex, direct experiences involving manipulation and sense perception become less necessary or feasible. Instead recourse must be taken to the use of pictures, diagrams, and other forms of visual perception such as are found in well prepared instructional materials or can be devised by the teacher.

The more able the student, the sooner he is able to shift from direct concrete experiences to abstract methods of thinking. The

[5] Van Engen, H. "The Formation of Concepts," *The Learning of Mathematics*, pp. 69–99. The Twenty-first Yearbook of the National Council of Teachers of Mathematics. Washington, D. C.: The Council, 1953.

higher the level of mathematical development the less possible and profitable it is to provide direct concrete experience. The efforts of teachers in the upper grades to include direct first-hand experiences are illustrated by field work in connection with the study of geometry.

Slow learners are weak in the use of abstract symbols and often are blocked because they have not had experiences that make new work meaningful to them. In spite of this fact instructional materials are often placed in the hands of these pupils that involve almost exclusively abstract language symbols. The problem of the slow learner can only be solved by the use of "concrete-action learning equipment." He needs the manipulative experiences out of which abstractions and concepts emerge. Later he can learn to interpret pictorial sequences and diagrammatic materials and thus become increasingly independent of concrete learning aids. The sequence of steps in the use of learning aids to develop meanings may be listed as follows:

1. Manipulation of concrete representative materials
2. Consideration of pictorial representations
3. Study of semi-concrete representations, such as diagrams, cut-out fractional parts, drawings, etc.
4. Study of symbolic representations, such as algorisms to be learned, explanations of procedures, etc.

Textbooks sometimes suggest the use of concrete materials which the teacher must provide to make number operations and concepts meaningful. Increasingly mathematics textbooks are using pictorial and semi-concrete representations in presenting new work. These aids help the student to understand the meaning of the symbolic or verbal description of the process to be learned. There can be no doubt about the value of such learning aids. Through their use the teacher can effectively integrate direct experience and vicarious experience involving the reading and study of printed materials.

Modern vs. Traditional Procedures

In much of the current literature dealing with the teaching of mathematics, emphasis is placed on the importance of making

what is being learned vital and meaningful to the learner. A number of important investigations[6] have shown that the learning of arithmetic is greatly facilitated when what is being learned is made mathematically meaningful and socially significant to the students. The principles underlying the procedures in arithmetic have important implications for all fields of mathematics.

Two theories of teaching arithmetic differing markedly in point of view underlie much of current practice in classroom instruction. The first of these may be called the modern approach; the second the traditional or drill approach. There are no standard definitions of either theory. There also is no general agreement among the sponsors of either theory as to the details of methods of applying them. However, a number of general principles are emerging which clearly indicate the differences between the two underlying points of view. In the following discussion, six contrasting principles will be presented. Some of the important evidence concerning the validity of these principles also will be summarized briefly. The consideration of ways of modifying current methods of teaching arithmetic and other branches of mathematics in the light of the two sets of principles should be helpful for all teachers.

The six contrasting principles underlying the two theories are:

CONTRASTING PRINCIPLES UNDERLYING TWO THEORIES OF THE TEACHING OF ARITHMETIC

Modern Approach	*Traditional Approach*
1. Learning takes place through experiences that are intrinsically, genuinely *purposeful*	1 *Extrinsic devices* are effective means for motivating learning.

[6] Brownell, W. A. *Learning the Multiplication Combinations* Durham, North Carolina Duke University Press, 1943

Brownell, W A. *Learning as Reorganization* Durham, North Carolina Duke University Press, 1935

Brownell, W A *Meaningful vs Mechanical Learning* Durham, North Carolina Duke University Press, 1949

Harap, H L. and Mapes, C. "Learning the Fundamentals in an Activity Program," *Elementary School Journal* 34 515–526

Thiele, C L *The Contribution of Generalization to the Learning of Addition Facts* Teachers College Contributions to Education, No 263 New York Bureau of Publications, Teachers College, Columbia University, 1938

2. Learning should be *meaningful* and induce *insight.*
3. *Discovery* of facts, meanings, and generalizations by the learner through inductive methods leads to understanding and insight.
4. Content should be so presented that the *perception of relations* is facilitated.
5. A wide *variety of learning experiences* should be provided to extend meanings and to assure needed practice.
6. Learning is a *growth process* leading gradually to responses at an increasingly mature level.

2. Learning is a *mechanistic neurological process.*
3. *Authoritative prescription* by the teacher through deductive procedures of the facts, ideas, and methods to be learned, assures correct connections.
4. Learning consists of the *forming of specific connections* presented as unrelated elements
5. A process of *repetitive drill* assures learning and mastery.
6. Performance *at the adult level* is expected and required at all stages of learning.

1. *Learning takes place through experiences that are intrinsically and genuinely purposeful.*

The learner should have meaningful worthwhile goals to guide his activities. Meaningful goals emerge from the desires, needs, dissatisfactions, and interests of the learner. They may emerge in any area of the curriculum. They may also grow out of values and needs emerging in life outside the school. Goals become meaningful to the learners when they participate in the setting up of their objectives and in planning and carrying out methods of achieving them. The efficiency of their work is greatly increased if they continuously evaluate and seek to improve their procedures. When learning is purposeful, all desirable aspects of the personality of the learner are developed, including his intellectual, social, emotional, even physical, traits. A realistic problem-solving situation undoubtedly is the most illuminating type of learning experience. Intrinsic forms of motivation such as the desire for new experiences, the desire for security, and the desire for recognition are operative in such activities.

2. *Learning should be meaningful and induce insight.*

The pupil should understand what he is learning. This applies to the aspects of arithmetic related to its mathematical phase as

well as to its social phase. Experiments have shown that understanding greatly aids learning and retention. Practice to develop skill should never be assigned until the operation to be practiced is meaningful to the learner and he understands the steps in the process. Simple methods of testing understanding have been devised that can be easily applied by any teacher.

The result of lack of understanding of a process is unintelligent manipulation of numbers and inability to recall what has been learned. The higher mental processes are not involved in such routine learning.

The contrast between the points of view of traditional and modern procedures with respect to this principle illustrates the fundamental difference between the two major psychological theories discussed above. According to the connection-forming theory, learning consists of the formation of connections through the well-known "stimulus-response" process, a mechanistic neural-path action, such as is the basis of most animal learning.

According to the field theory, all elements in the total learning situation are utilized that may aid the learner to understand what is being taught, especially those manipulative, visual, and verbal aids that will make his work meaningful and sensible to him.

3. *The discovery of facts, meanings, and generalizations by the learner through inductive procedures leads to understanding and insight.*

Through the use of exploratory materials and visual aids, as will be shown in the following chapters, the students can be led to discover independently many quantitative facts, relations, and generalizations. Thus by manipulating fractional parts of wholes, they can quickly discover many relations among these parts. Discovery leads to insight.

4. *Content should be so presented that the perception of relations is facilitated.*

Arithmetic should be taught in such a way that the learner constantly reorganizes and structures what he is learning. The teacher can assist this process by leading the pupil to make generalizations about what he is learning. For example, the

learner can be shown that addition and multiplication are related operations. Similarly, he can discover that the products of the fives all end in 0 and 5, a valuable aid to learning these facts. Hundreds of similar generalizations can be developed.

The best single illustration of the operation of this principle is the relative ease of memorizing meaningful structured material such as a poem in which the organized relationships among ideas are evident to the learner, as against the difficulty of memorizing a group of nonsense syllables of equal length. Experiments have abundantly demonstrated the relative ease of memorizing meaningful organized material and the value of assisting the learner to reorganize at higher levels what he is learning.

5. *A wide variety of learning experiences should be provided to extend meanings and to assure needed practice.*

Learning takes place most efficiently and easily when the arithmetic being learned is experienced in vivid, realistic, meaningful situations. These experiences may range from the manipulation of concrete objects and the use of other visual aids to get meanings to activities requiring the uses of number and quantitative procedures in dealing with genuine personal and social problems. The wide range of activities possible enables the teacher to adapt instruction to differences in the rates and ways in which children learn. Emphasis is placed on learning skills and concepts in functional situations. In this way, the mathematical and social phases of arithmetic are both made meaningful to the learner.

Under the modern approach, the classroom is organized as a learning laboratory in which are provided many and varied aids to learning adapted to the needs of children of various levels of ability. Many experiences are arranged to stimulate, enrich, and vitalize learning. Such a program is in sharp contrast with the drab, narrowly conceived activities of classrooms in which the work in arithmetic consists chiefly of long periods of repetitive drill devoted to the mastery of what too often are meaningless processes whose functional value the pupils often do not realize. Repetition without intent to learn never leads to genuine learning.

6. *Learning is a growth process leading gradually to responses of an increasingly mature level.*

It is generally recognized that there are large differences in the rates at which children learn. It has been shown that over a period of time and under good guidance their responses will change from the immature, fumbling reactions that are found during the initial stages of learning to the mature methods of thinking that the school wishes the students to master. From grade to grade the teacher faces the problem of assisting them to discard immature thought processes in favor of more efficient procedures. Roundabout procedures are symptoms of immaturity, not necessarily of poor ability.

General Comment about the Application of These Principles

It should be stated that at the present time, we rarely find instructional programs that apply fully and efficiently either of the groups of six contrasting principles that have been discussed. Furthermore, evidence is lacking as to the complete effectiveness of instruction organized according to either theory. However, a mass of information is accumulating about the relative effectiveness of the meaningful teaching of arithmetic over mechanized drill. Brownell and Moser's[7] recent study of the effectiveness of different methods of teaching "borrowing" in subtraction illustrates this point.

Every effort should be made to discover efficient methods of applying the principles of the "meaning" theory to the teaching of arithmetic and other branches of mathematics. The cumulative evidence in favor[8] of the meaning theory should lead all teachers to utilize methods of instruction that will make arithmetic meaningful to all learners. Every effort should be made by all teachers of mathematics to invent methods and materials of instruction that will assist the learner to *understand what he is*

[7] Brownell, W. A. and Moser, H. E. *Meaningful vs. Mechanical Learning: A study of Grade III Subtraction.* Durham, North Carolina: Duke University Press, 1949.

[8] The reader should consult the volume by the authors, *Making Arithmetic Meaningful*, for a comprehensive treatment of methods of making number meaningful to children in the elementary grades. Research is systematically reviewed there.

learning. This is the key to the successful teaching of arithmetic. It is the heart of the modern approach which the authors stress throughout this book.

Levels of Learning

The concept of levels of learning which is repeatedly stressed in this book recognizes the fact that children in almost any class, even one organized on the basis of the mental ability of the students, do not all learn in the same way and almost always learn on several different levels of maturity. To illustrate, some of the more mature students will understand the algorism for the subtraction example shown abstractly at the right; others will need diagrams or pictures to help them visualize the transformation of 2 to $1\frac{2}{2}$; others will find it necessary to work with manipulative materials such as objects to be cut or fractional parts in a fraction kit before they are able to perceive the transformation shown in the algorism. Obviously the first of these three procedures is the most mature and indicates the ability to operate at a high level of abstract thinking; at the second level the use of visual and pictorial aids is necessary,

$$\begin{array}{rcr} 2 & = & 1\frac{2}{2} \\ -\ \frac{1}{2} & = & \frac{1}{2} \\ \hline & & 1\frac{1}{2} \end{array}$$

Different devices for the study of fractions represent different aspects of the third level of operation.

Public Schools, Portland, Oregon

clearly a less mature level of learning than the first; at the third or lowest level it is necessary to go to the level of concrete experience. In mixed classes all levels of learning will be found.

After a preliminary presentation of a topic, the teacher should divide the students into groups according to evident differences in levels of thinking and learning, and then proceed along lines suggested by the needs of the students. At least two major groups should be formed: those who can proceed abstractly and are ready for systematic practice, and those who can proceed only with the help of visual aids. The latter group is not yet ready for abstract procedures. There may be some slow learners who may even need help to do the work objectively. The teacher should adapt the pace of instruction to the rates at which the individual students learn, so that finally all can proceed successfully at a relatively high level of learning.[9]

The Discovery of Relationships and Generalizations

The discovery of relationships among processes is another valuable aid for the extending and enriching of mathematical meanings and for raising the level of learning of students, as indicated in the following statement:

> (a) Addition and subtraction are opposite processes—putting together and taking apart. (b) Multiplication and division are opposite processes—putting together and taking apart. (c) Multiplication is usually a faster putting-together process than addition. (d) Multiplication always deals with equal-sized groups, while addition may deal with either equal-sized or unequal-sized subgroups. (e) Division is usually a faster taking-apart process than subtraction. (f) Division always deals with equal-sized groups, while subtraction may deal with either equal- or unequal-sized subgroups.[10]

[9] For a longer discussion of the concept levels of learning as applied to arithmetic, consult Smith, R. R. "Provisions for Individual Differences," *The Learning of Mathematics—Its Theory and Practice*, Chapter 10. The Twenty-first Yearbook of the National Council of Teachers of Mathematics, 1953. The Council, Washington, D. C.

[10] Swenson, Esther J. "Arithmetic for Preschool and Primary-grade Children," *The Teaching of Arithmetic*, p. 66 Fiftieth Yearbook of the National Society for the Study of Education, Part II. Chicago: University of Chicago Press, 1951.

When the learner knows the meaning of an operation and understands the processes involved in working a new kind of example in the operation, we can say that he has in fact learned it. However, practice is needed to broaden its meanings and to develop proficiency in using it. The actual application of the new step in a variety of situations and in different contexts is one valuable form of practice. Then the work is well motivated and the students see the usefulness of what they are learning.

Systematic practice with specially prepared materials, sometimes called "drill," is an essential form of practice that provides the recurrent experiences that are necessary to develop speed and proficiency in computational skills. Mixed drills spread over a period of time are an efficient means for maintaining skills.

Research has shown that repetitive drill without understanding as used in many schools is a wasteful procedure, actually detrimental to learning. Under such conditions pupils engage in a routine mental exercise limited to the mastery of a body of specific skills taught without regard to their meanings, to the relationships among the various number processes, or to their uses in everyday life.

The organization of practice to develop and perfect skills can be greatly improved by applying a few simple principles as:

1. Skills are not isolated mechanisms which can be learned apart from real situations in which they function and then be automatically applied in real situations. When meanings and relationships have been established, these meanings and relationships can be extended and perfected through practice. Skills are learned better when they are closely related to meanings and when the learner sees how he will use them. The greater the variety of situations in which skills are used, the better they will be learned.

2. Skills are not learned readily in advance of need or of meaning and familiarity. A period of integrative practice during which the learner becomes familiar with the new step and gains an understanding of it should precede the systematic practice that is needed to develop proficiency and skill.

3. Skills are not fixed, unchanging methods of response that may be acquired as such through unthinking repetition. They gradually change from the slow, immature methods of response characteristic of immaturity to the rapid, efficient procedures and thought processes that approach those used by adults. Skill performance varies from person to person and from situation to situation, depending on the conditions under which the skill is used. To illustrate, we proceed slowly and carefully when accuracy is essential. On the other hand, when we wish to make an approximation, we proceed more rapidly and with less stress on exactness.

4. In the initial stages of practice considerable emphasis should be placed on diagnosis. The teacher should observe the pupil's methods of work and the thought processes he uses. Every effort should be made to lead the pupil to utilize increasingly mature methods of thinking. Considerable stress should be placed on self-directed practice so that the pupil will be obliged to take a growing responsibility for the improvement of his skills. Practice periods should be relatively short and distributed over a span of time. Practice is likely to be more fruitful when the pupil is aware that he is making progress.

5. Students who have practiced certain skills for which they have a genuine need are likely to regard similar drill on other functional skills as necessary and sensible. Routine practice alone is not enough; it is important that practice be made purposeful by setting up goals that are meaningful and achievable. The systematic practice needed to establish skill can be done with suitable types of materials during special blocks of time set aside for this purpose.

6. To maintain arithmetic skills it is necessary to provide distributed practice which should be spread out over a period of time. This practice should be cumulative in nature, consisting of mixed types of examples in a variety of processes so that the pupil must shift from one operation to another as he works the examples. Many modern textbooks provide a suitable maintenance program of mixed practice. If this type of distributed practice is not provided, there is likely to be a marked deterioration of skills because the activities of the school provide relatively

few opportunities for the pupils to apply some of the arithmetic processes they are required to learn. However, forgetting will be reduced to a minimum when learning has been meaningful and functional and when there is insight into the operations.

7. Suitable provisions should be included for helping the learner to see the progress he is making and to locate difficulties and weak spots. The practice materials should be interesting and well written; they should be organized in such a way that the pupil can progress at his own rate with a minimum amount of guidance from the teacher. The use of materials prepared by specialists is recommended. Reasonable standards of achievement should be set up which are adapted to the ability of the learner and his level and rate of development, so that overlearning will not be carried to unjustifiable extremes. The concept of repetitive practice as here used can be extended beyond the learning of computational procedures to include the development of interests, attitudes, appreciations, understandings, purposes, and social insight. Instruction in mathematics also can be organized so as to provide practice in cooperation, leadership, and other forms of desirable social behavior.

Teaching Reading Skills Needed in Arithmetic

Success in arithmetic and other branches of mathematics is closely associated with proficiency in the special types of reading skills necessary in this field. This is particularly true in the reading of explanations of procedures given in textbooks, in problem solving, and in the gathering of information of materials from printed sources for any number of different purposes. Special consideration is given to the development of reading skills in Chapter 9.

e. Provision for Individual Differences

The Nature of Individual Differences

The most perplexing and difficult problem faced by the teacher is that of providing for the wide range of differences among the

members of the class, from whatever point of view the group may be considered: their ability to learn mathematics, the rates at which they learn, the level to which skills have been developed, their interests, their attitudes toward the subject, their background of experiences, their social status, and so on. Sometimes the problem is complicated by conditions associated with mental hygiene and the relationships of individuals to each other and to the group as a whole.

The teacher should bear in mind that in the study of large units of work dealing with the social phase of mathematics the class can work as a single group. However, the work within the class can be so organized that each student will have the opportunity to contribute to the whole undertaking in terms of his interests, abilities, and special talents.[11] This is a very effective method of providing for individual differences within any group; no student need be expected to work alone, as an isolated individual. He will grow more from every point of view if he makes his unique contribution as a member of the whole group. Chapter 8 contains a detailed discussion of this problem.

Providing for Individual Differences in the Presentation of Skills

In the typical class, pupils will vary greatly in their need for the use of the different forms of presentation of new work related to mathematical operations, or in remedial instruction. When all types of materials are used, as discussed on pages 79 to 80, the teacher can be sure that instructional procedures are providing effectively for differences in the ways in which the different pupils learn. The slower learners usually are aided by the sequential uses of manipulative or exploratory materials, and pictorial and visual aids, which are finally followed by the discussion of the symbolic algorism to be learned. The more able students perhaps do not require as detailed help and in some cases can get the idea very readily from a well-written textbook presentation. However, even they will profit from the complete presentation we have outlined. The teacher can then give a short preliminary

[11] Much valuable information may be found in the bulletin, *Curriculum Materials in High School Mathematics*, Bulletin 1954, No 9 Washington, D. C U S. Department of Health, Education and Welfare, 1954.

test to determine how well the new step is understood by the different members of a class.

On the basis of the results of this test, the teacher can group the students according to their needs. Those whose papers indicate that the step has been learned can be assigned practice to establish the procedure. When there is evidence of difficulty or lack of understanding in the work of any students, the teacher should diagnose the difficulty, and then reteach as may be necessary, using any materials that will lead to understanding by the students, even simpler materials than those used in the original presentation. Procedures for diagnosing learning difficulties are discussed in Chapter 13 and also in the various chapters dealing with methods of teaching in specific areas.

Curriculum and Instructional Adjustments Possible

It is important that in the typical class the teacher adjust the content of the curriculum, the instructional procedures used, and the standards to be achieved to the differences in the abilities of the children. It seems reasonable that the work of slower pupils should not be rated on the same basis as the work of superior students.

A helpful analysis of the wide variety of procedures that are used in our schools to adapt instruction to individual differences in the ability to learn mathematics and other fields is given below.

1. The use of experience units which provide for a wide variety of activities at different levels of difficulty (For details see Chapter 8.)
   a. Problem solving, research, and experimentation
   b. Construction activities resulting in intellectual or material products
   c. Appreciation experiences enjoyed by the individual
   d. Creative activities resulting in original thinking, acting, and producing
   e. Excursions, field trips, and participation in community enterprises
   f. Opportunities for learning through use and direct experience.
2. Grouping of pupils according to their needs, interests, and level of development.
   a. Classification into groups of similar social maturity and intellectual status

b. Promotion at irregular intervals
c. Program planned in terms of future needs of individual
d. Exploratory courses
e. Classes for gifted children
f. Special provisions for talented children to insure stimulating experiences
g. Rich program of co-curricular activities.

3. Differentiation of work within classes.
a. Adapting program of work to level of pupil ability
b. Readiness programs adjusted to needs of individuals and groups
c. Differentiated assignments
d. Differentiated standards to be achieved
e. Differences in scope of course requirements
f. Differences in time allowed for completing work
g. Supplementary assignments
h. Special assignments for more able pupils or those with special interests
i. Use of books and materials of several levels of difficulty
j. Use of workbooks.

4. Laboratory methods.
a. Individualized instructional materials to develop basic skills, such as those used in the Winnetka plan
b. Dalton plan of assignments of different levels of difficulty and comprehensiveness
c. Morrison plan of guide sheets and differentiated assignments
d. Individual progress plans in laboratory and shop courses
e. Diagnosis of difficulties that arise in the course of learning
f. Remedial and corrective measures to eliminate causes of difficulty
g. Provision of a wide variety of materials for developing meanings
h. Use of community resources to vitalize and enrich learning experiences.

5. Special provisions for maladjusted and slow-learning pupils.
a. Adjustment and coaching teachers
b. Opportunity classes
c. Ungraded classes
d. Hospital classes for serious problem cases
e. Special classes for students who have failed some required courses.

6. Guidance services which assist in orienting the student and in planning a program of work adjusted to his needs, interests, and potentialities.
a. School psychologists
b. Visiting teachers and social workers

c. Counselors and vocational guidance experts
d. Home-room teachers and advisory periods
e. Medical services
f. Clinicians to study behavior problems and cases of serious retardation.

Methods of Socializing the Learning of Mathematics

The values of socializing mathematics that were recognized by selected junior high school teachers in 635 schools are indicated by the results of a recent study.[12] These teachers were rated as highly efficient by their principals in dealing with rapid and slow learners. The data show that among thirty instructional procedures investigated, the following were reported as among those used most frequently for rapid and slow learners:

	Rank	
	Rapid	*Slow*
1. Assist students in learning vocabulary and reading skills peculiar to mathematics	1	3
2. Emphasize social uses of mathematics	2	4
3. Encourage study of the applications of mathematics to science	4	15
4. Encourage solution of mathematical problems from field of students' interests	10	13
5. Give to students experience in applying the principles of mathematical reasoning to social problems	16	18

On the other hand the following of the thirty procedures listed were those that were reported as being used most infrequently with both rapid and slow learners:

	Rank	
	Rapid	*Slow*
1. Provide field trips related to classwork	28	28
2. Provide mathematics laboratory	29	29
3. Provide students with experiences in a Mathematics Club	30	30

[12] *Teaching Rapid and Slow Learners in High School*. Bulletin 1954, No. 5. Pp. 38 and 50. U. S. Department of Health, Education, and Welfare. Office of Education, Washington, D. C.

Less frequently used socializing experiences were the following:

	Rank Rapid	Rank Slow
1. Encourage students to make up problems by securing data from own reading and experiment	18	23
2. Provide students with experience in evaluating types of reasoning in newspaper and magazine articles	21.5	22
3. Encourage students to make scrapbooks and prepare graphic materials showing uses of mathematics	24	20.5
4. Encourage students to read stories about mathematics or famous mathematicians	25	25

Activities Suitable for More Able Students

Mathematics teachers should plan to explore and develop the interests and aptitudes of all students, especially those who have unusual capacity and talent in this field. To this end teachers

The use of a flexible model can lead to many "discoveries" on the part of the learner.

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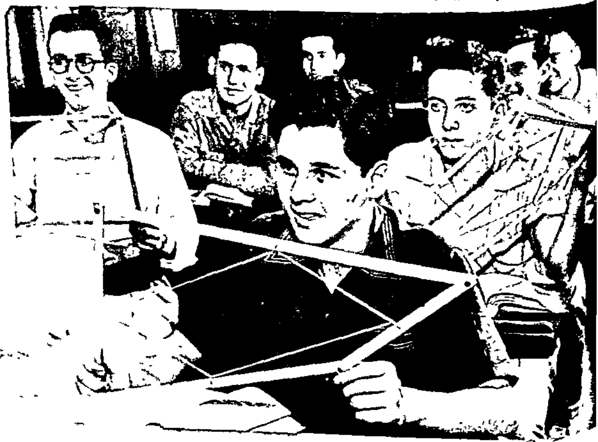

should give them the opportunity to engage in a wide variety of interesting, challenging activities that will enrich their learning experiences and stimulate them to carry on independent studies. The following list suggests some of the more valuable procedures that teachers have used successfully to make adequate provision for these more able students:

1. Exploration of the more difficult application of topics and processes being studied
2. Extension of mathematical principles beyond the limits of the common essentials
3. Work with mathematical puzzles
4. Solution or development of magic squares
5. Preparation of reports on assigned or pupil-selected topics
6. Dramatization of important social applications of arithmetic
7. Experiments with numbers, shortcut methods, etc.
8. Encouraging the discovery of procedures
9. Excursions to banks, stores, etc.
10. Field work, such as laying out gardens, athletic grounds, etc.
11. Construction projects
12. Drawing of interesting geometric designs
13. Preparation of models, exhibits, etc.
14. Collecting pictures to illustrate applications of number
15. Keeping mathematics notebooks or scrapbooks
16. Conducting school bank, school store, etc.
17. Debates on current issues and problems
18. References to books in which the history and present status of measurement and similar quantitative topics are discussed in interesting ways
19. Mathematics clubs and similar interest groups
20. Considering recent developments in quantitative procedures
21. The encouragement of original methods of attack on quantitative elements of social situations
22. Having the better pupils assist slower pupils or others who are encountering difficulties.

There is an extended discussion of ways of enriching instruction for gifted learners in Chapter 14.

Questions, Problems, and Topics for Discussion

1. What are some of the basic principles that are discussed in this chapter? Which do you regard as the most important?

2. Observe a mathematics lesson. Look for illustrations of the kinds of outcomes listed on page 60. Are there any important outcomes that were overlooked by the teacher?

3. Why is a problem solving situation such a valuable type of learning situation? Illustrate.

4. How can the teacher discover problems that are of concern to the students? Which are more fundamental, personal problems or problems of broader social significance?

5. Why is mathematics so important in the study and solution of problems? Discuss and illustrate the seven major concepts underlying problem solving that were discussed in the report, "Mathematics in General Education."

6. How can the study of mathematical operations be approached on a problem solving basis?

7. Is there any difference between "quantitative thinking" and "problem solving"? What is meant by "functional"?

8. What are the differences between "connectionism" and "field theories"? How do they affect curriculum and method?

9. What are "meanings"? What kinds of "learnings" are there? When is learning "meaningful"?

10. Discuss the six contrasting principles presented on pages 76–77. Show the differences between the modern and traditional theories by giving specific illustrations of procedures that bring out the contrasts.

11. Observe some lesson and apply these principles to the instructional procedures used.

12. Prepare a lesson plan showing how to apply modern principles in the teaching of some unit of work.

13. Illustrate the "levels of learning" discussed in this chapter. Observe a group of students in a lesson and try to discover illustrations of each level.

14. What are "generalizations"? Give examples. Why stress them?

15. What are "relationships"? Illustrate.

16. Some specialists differentiate between learning by use, by practice, and by drill. Can you see any differences in these concepts? What is the place of practice in learning?

17. What reading skills are involved in the study of mathematics?

18. What procedures can teachers use to adapt instruction to individual differences? In what ways do individuals differ? Do individuals have common needs in mathematics related to the mathematical and social phases of the subject?

19. Study carefully the list of methods of providing for individual differences given on pages 87–89. Select those that you think are most useful and practical. Study practices in local schools to see what is done for superior students.

Suggested Readings

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Chapter 4

Materials for Equipping the Mathematics Classroom

This chapter deals with the following topics:

a. The function of the classroom
b. Exploratory materials
c. Visual materials
d. Symbolic materials.

a. The Function of the Classroom

The Classroom as a Learning Laboratory

The equipment needed in a mathematics classroom depends upon its function. If its function is solely to provide a place for a student to work examples and to solve verbal problems as given in a textbook or a workbook, the needed equipment would consist primarily of paper, pencil, and a textbook or a workbook. In this case the classroom is a place for learning to perform computations. On the other hand, if the classroom is to be a place for discovering meanings and procedures and for experiencing directed learning activities, adequate equipment must be provided for meeting these needs. Then the classroom becomes a workshop or a learning laboratory equipped for teaching mathematics comparable to a laboratory equipped for teaching the social studies or the physical sciences.

It is important for the teacher to understand that a classroom adequately equipped as a laboratory does not guarantee that a student will learn mathematics meaningfully. He may manipulate the materials provided but not profit greatly from the experience. In this case the classroom is equipped with gadgets which are used predominantly for activity purposes but not as learning aids in discovering number relationships. The way in which materials are used determines whether or not the student will profit from the experience in dealing with them. The thought pattern experienced by the student when he uses an object differentiates *exploratory material* from *manipulative material.* If a student uses fractional cut-outs to discover a principle which he could not understand through a symbolic presentation, the cut-outs serve as exploratory material. On the other hand, if the student uses the cut-outs in a perfunctory manner to represent fractions, these materials become predominantly manipulative. The value derived from equipping the classroom as a laboratory depends upon the way the teacher has the student use the material. The learner may merely manipulate the material as an activity or the teacher may teach the student how to use it to discover relationships among quantities. The primary function of different kinds of materials in mathematics is to enable the student to understand each step in an operation. Obviously, the experiences the student has had in dealing with various types of materials will determine the particular type needed for his level of operation.

The Process of Learning Is Vital

Purdy and Kinney stressed the fact that it is the *process* of learning instead of the *product* of learning which is vital. Many teachers evaluate learning in terms of the end product and give almost no consideration to the means the student used to attain that end. This procedure does not give adequate emphasis to understanding in learning. If the process of learning is such a vital element in an instructional program, the student should use a variety of materials to acquire an understanding of a process. These authors stated that " . . . good teaching is effective

guidance of this process [learning]. This process is continuous, and the individual is usually at various levels with respect to several learnings at once, many of which never develop into generalizations. For purposes of discussion, however, we see a given learning experience developing somewhat as follows:

1. A problem situation which the pupil is ready to attack	→	2. Solution, through exploration and discovery	→	3. Abstraction, symbolism, generalization	→	4. Application (transfer)

. . . The *process* of problem solving, until it has been perfected, must be guided at each level."[1] The second step in the learning process calls for the use of representative manipulative or exploratory materials.

Kinds of Materials

Three kinds of materials are needed to teach mathematics effectively in the junior high school. These materials may be classified as *exploratory*, *visual*, and *symbolic*. Exploratory materials are those that the student can touch, move, or manipulate, such as an abacus or a ruler. As previously stated, it is the use of objective material which makes it exploratory instead of purely manipulative. Charts, posters, pictures, films, and slides are representative of visual materials. A page of explanations of operations or of examples or problems in a textbook or workbook is representative of symbolic materials.

It is important for the teacher of mathematics to plan a laboratory of suitable materials for teaching the subject. Too much emphasis should not be placed on the use of symbolic materials, such as the textbook, in preference to other types of materials. The proper functioning of each kind of material in connection with learning experiences of the student is essential to an effective program for teaching elementary mathematics at

[1] Purdy, C. Richard and Kinney, Lucien B "Directing Learning in Arithmetic," *Elementary School Journal*. 54:288–89. Chicago: University of Chicago Press.

any given age or grade level. As the pupil grows in his ability to deal intelligently with quantities, he should be able to show a corresponding growth in ability to deal with symbolic materials. If he is able to understand the work with symbols, he need not use either visual aids or exploratory materials. Good teaching is in evidence *when the student is challenged to work at the highest level of abstraction at which he understands the work.* Conversely, poor teaching is in evidence when a student is permitted to use materials that are adapted to either a lower or a higher level of abstraction than the level at which he can understand the work. The kind of material which a student should use while learning about a given process or topic must be determined by the teacher through observation and analysis of the student's work.

b. Exploratory Materials

Types of Exploratory Materials

In many schools the work in the upper grades is departmentalized. We may assume that the teacher of mathematics at this level teaches no other subject. Generally, the teacher of mathematics uses one particular classroom. If the teacher must use several different classrooms during the school day, the problem of equipping each classroom is much more difficult than when the teacher uses only one room.

There are two types of exploratory materials, depending upon their usage. One type is the kind of material used by the student that we shall designate as *student-discovery* material, and the other type used predominantly by the teacher we shall designate as *teacher-demonstration* material. The teacher-demonstration material must be larger in size than the student-discovery material so that the material used in a class demonstration can be seen from all parts of the classroom.

One of the basic problems to be considered in equipping the classroom is the determination of a minimum list of materials. It is better to have in the classroom a few materials which are used effectively than to have the room filled with a wide variety

of materials that are seldom used. Often the teacher is confronted with the problem of providing suitable space for storing laboratory equipment. If the material must be stored in boxes which frequently are put in inaccessible places, the material will seldom be used. The number of different kinds of exploratory materials which a teacher should have in the classroom depends upon the facilities available for using and storing the material. A fully equipped laboratory should have available suitable space for storing many more kinds of exploratory materials than are needed for a classroom which contains only the basic essentials for teaching the subject.

A Table for Demonstration Purposes

The first essential piece of equipment in the arithmetic classroom is a table which is used for demonstration purposes. The top of the table should be approximately 32″ × 42″ and its height should be about 30 inches. In many classrooms the top of the teacher's desk is the only place available for exhibiting teacher-demonstration material. The desk usually contains the teacher's books and other materials so that it is not suitable for displaying demonstration materials. If a teacher does not have a desirable place, such as a table, for displaying exploratory materials, very probably these materials will seldom be used. The classroom must be equipped with a table if such material as a flannel board or a hundred board is to be used effectively. The table serves both for demonstration purposes and for displaying materials.

A Student's Kit

Ideally, each student should be provided with a kit of materials. The materials used in teacher demonstrations should be matched by the student's materials. As students discover how to add two fractions by use of fractional cut-outs, the teacher should use fractional cut-outs to demonstrate the procedure on a flannel board. The students should explore different ways of finding the answer to a given example. Then the teacher should give a

demonstration of the best way to perform the operation. The method demonstrated by the teacher is the algorism which the race has found to be the most effective way of performing a given process.

If a classroom has a wide variety of exploratory materials, a student's kit need not be supplied with all of the corresponding materials. The kit should contain those materials which will enable the student to discover the procedures in the basic work for a given year. Chapter 5 describes the kind of material the student should have in his kit to deal with integers. Chapter 6 discusses the kinds of materials a student should use to deal with common and decimal fractions. The items for a student's kit for mathematics at the junior high school level should consist of: (1) fractional cut-outs; (2) squares and rectangular strips; and (3) compasses, protractor, and ruler. Pages 146–147 give a description of the squares and rectangular strips needed for objectifying the work dealing with integers and decimal fractions. The fractional cut-outs may be used for dealing with common fractions. The items in the third group are necessary when dealing with informal geometry in the junior high school. It is assumed that the teacher will supply the student with cross-ruled paper for drawing graphs.

The three different materials, mentioned above, are adequate for the student to discover most of the mathematical procedures which can be objectified except those dealing with per cent. Chapter 7 shows that a hundred board is an effective teaching aid to use to help the student to understand the meaning of per cent. It is possible to make a similar teaching aid by using a piece of cardboard divided into small squares. A cross-ruled square on the chalkboard divided to show 100 squares could be used as a visual aid in the same manner as the divided cardboard.

Students who understand the work with integers and fractions when dealing with symbols do not need objective materials. In the upper grades there will be very few students who understand the work with decimal fractions so well that exploratory materials will not be helpful in enriching the meaning of the topic in some way.

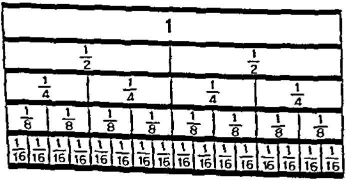
A Teacher's Kit

The teacher should have a few basic materials for demonstration purposes just as a student should have certain exploratory materials for his kit. The classroom should contain many other items in addition to the basic materials for the teacher's kit. The minimum list of items for teacher-demonstration purposes includes: (1) a flannel board with fractional cut-outs; (2) a fraction chart; (3) place-value pockets; (4) an abacus; and (5) a hundred board.

The five pieces of demonstration material should enable the teacher to objectify most of the work dealing with integers, common and decimal fractions, and per cent. Squares, approximately 2 inches on a side, covered with flannel or made of felt may be used on the flannel board for teaching the concept of area. As will be shown, additional materials will be needed for the teaching of informal geometry.

A teacher can make a *flannel board* by covering one side of a piece of plywood or other rigid material, approximately 20" × 30", with flannel which contains a good nap, or with doeskin cloth. The circles for the cut-outs should be approximately 10 inches in diameter and covered on both sides with flannel.

The illustration below shows a *fraction chart* containing five parallel sections of the same size. Five sections are needed to show a whole, and wholes divided into halves, fourths, eighths, and sixteenths. The teacher should also have strips showing wholes divided into thirds, sixths, twelfths, and tenths.



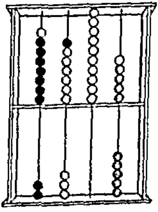
It is easy to make a place-value pocket chart from tough cover paper or tag board, approximately 20″ × 26″ in size. Place the paper on a table with the short edges at top and bottom. Measure down $5\frac{1}{2}$ inches from the top edge of the paper and draw a line parallel to this edge. Fold the top along the line drawn and crease firmly. Then, measure down $2\frac{1}{2}$ inches from the crease and fold under. Open the chart flat on the table. Lift the lower creased fold up over the upper crease. The upper crease will slide down under the fold and will form the first pocket. Crease firmly and staple each edge of the pocket.

To make the second pocket, measure down 4 inches from the edge of the first pocket and crease. Measure down $2\frac{1}{2}$ inches and crease. Repeat for the third pocket.

Divide the sheet longitudinally so as to form three equal pockets in each row. At the points of intersection with the rows of pockets, insert staples. Then letter the top of the sheet to represent ones, tens, and hundreds. The chart should be fastened to a wire coat hanger so that it will be possible to display the chart in a vertical position at the front of the room.

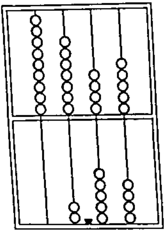
To make the cards for the pockets, use oak tag of a different color from the chart. Each card should be approximately $1\frac{1}{2}$″ × $3\frac{1}{2}$″.

A more stable pocket chart than one made of cardboard may be made of wood. The most useful place-value chart for demonstration purposes should contain three rows of pockets. Then a complete algorism can be objectified in most processes. Markers for the pockets may be made of oak-tag or of tongue depressors.



An abacus can be used to objectify the decimal relationship of our number system and to show why zero is used to hold an empty place in our number system. To show that 1 bead on a rod has a value equal to all the beads on the next rod to the right, an abacus should contain 10 beads on a rod. On each rod on the abacus shown, there are 10 beads. Nine beads

are of the same color, which differ from rod to rod. The tenth bead is of the same color as the 9 beads on the next rod to the left. A rod without beads holds an empty place on an abacus just as a zero holds an empty place in a number. The number represented on the abacus is 2305. However, an abacus used to represent a number need contain only 9 beads on each rod. The 9 beads are then analogous to the 9 digits.



It is possible to represent a decimal by inserting a peg, to be used as a decimal point, in a hole bored between any two rods in the horizontal frame of the abacus. A small bottle cork inserted in the hole makes the represented decimal point visible to the class, thus identifying ones' place. The cork in the abacus on the left shows that the rod to represent ones' place is the third rod from the right. The number represented on the abacus is 2.54. The abacus is very effective when used for demonstrating that 1 one is equal to 10 tenths, or 1 = 1.0, or that .1 = .10.

The hundred board is a board which is enclosed in a square frame. The board contains 10 rows of disks with 10 disks per row. Each disk, made of cardboard, is about 3 inches in diameter. The disks stay in place when the board is in a horizontal position or when the board makes an angle of not more than 30° with the top of the table. If a board contains 100 pegs and the disks are modeled in the shape of a washer so that they will stay on the pegs, the board can be placed in a vertical position. Then it would be visible from any part of the classroom.

Demonstration Materials for the Classroom

There can be no standard list of materials for equipping the classroom for teaching mathematics at the junior high school level. It is possible, however, to suggest certain exploratory materials which may be used effectively in teaching different

topics. In the next few pages, different materials are suggested which the teacher should find useful in teaching certain topics. Those listed are not the only materials that can be used. The articles listed are effective teaching aids which do not cost much whether they are purchased from commercial distributors or student made. The teacher should always be alert to discover new materials with which to present, illustrate, or objectify topics. The teacher should be cautioned, however, to keep in mind that the value of any manipulative material is measured by its *effectiveness in helping the student to clarify concepts and to discover relationships among quantities*, and not by its novelty or uniqueness.

1. *Materials for Teaching the Structure of the Number System*

The following materials are helpful in showing the structure of the number system:

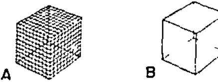
a. A place-value chart with markers

b. An abacus

c. A set of blocks, sticks, and slabs divided decimally:

1 hollow cube 10 inches on a side	1 slab containing 10 1-in. strips	1 10-inch stick an inch square on end	1-inch cube

Three of the faces of the large hollow cube should be ruled as shown in A and the other three faces should not be ruled, as shown in B. Then it is possible to use the cube to represent both integers and decimals. The cube in A represents 1000. When the unruled faces are shown, the cube represents 1. Then the component parts of the cube represent .1, .01, and .001, respectively.



A B

Our money system is decimal in form. At the junior high school level, it should not be necessary to use money to show the meaning of the decimal system of number.

2. *Materials for Teaching the Basic Processes with Integers*

The student has his kit of materials for discovering the procedures to follow in performing the different processes. The material which the teacher needs for demonstration purposes should match the material the student uses. The essential teaching material for objectifying the four basic processes with integers is a place-value chart with markers. (See Chapter 5.)

3. *Materials for Teaching Common and Decimal Fractions*

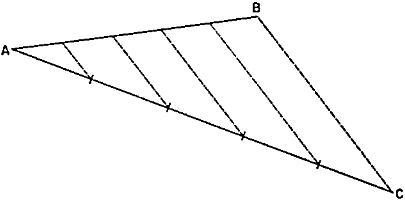
There is a wide variety of material which the teacher may use for objectifying the meaning of both common and decimal fractions. Some of the most effective materials for this purpose include the following:

a. Flannel board with fractional parts
b. A fractional chart for both common and decimal fractions
c. Disks, approximately 12 inches in diameter, cut from plywood about $\frac{3}{8}$ inch in thickness and divided to show halves, thirds, fourths, sixths, and eighths.
d. A foot ruler graduated to sixteenths of an inch
e. A foot ruler graduated to hundredths
f. A surveyor's tape which is graduated to hundredths of a foot
g. A meter stick graduated to millimeters and centimeters
h. A speedometer having a trip dial which can be set at zero
i. A place-value chart as used for objectifying integers
j. A micrometer and a steel rod approximately a half inch in diameter
k. The set of blocks, slabs, and sticks used to demonstrate the decimal relationship with integers may be used to show the same relationship with decimals.

The fractional parts for the flannel board should be sectors of circles approximately 10 inches in diameter. They can be covered with either flannel or doeskin.

The fractional disks made of plywood should be cut into fractional parts to parallel the fractional parts of the flannel board and of the student's kit.

Students in the junior high school should use compasses and straightedge to find a tenth of a foot, the unit for dividing a foot ruler into 10 equal parts. This length can be found by dividing a line 6 inches long into 5 equal parts. In the figure,



AB represents a length of 6 inches. AC is any intersecting line on which the student marks off five equal lengths, such as 2 inches, and then connects the points C and B. At each point of division on AC, parallels are constructed to BC. The points at which these parallels cut AB should be one-tenth of a foot apart. At this stage no proof is given for the construction. On the other hand, the construction should give the student a background for understanding one of the theorems taught in demonstrative geometry which states that if parallels intercept equal lengths on one transversal, they intercept equal lengths on every transversal.

To be useful the speedometer must have a trip dial and a stem to operate this dial. Then it is possible to set the dial at zero. By turning the stem governing the trip dial, the student can show that 10 tenths are equal to 1 whole.

A place-value chart must be properly labeled when objectifying decimals. Place-value charts may be made of wood. A usable form of place-value chart contains three rows of pockets on one side and three rows of pockets for decimals on the reverse side.

One of the most effective teaching aids a teacher can have is a combination flannel board and place-value chart. One side of the board should be covered with flannel. The other side should contain three rows of pockets about $1\frac{1}{2}$ inches deep. The cardboard labels on the pockets should be adjustable for both integers and decimals.

4. *Materials for Teaching Percentage*

The number of different kinds of manipulative materials for teaching percentage is much less than the number of types used for teaching most of the other topics in arithmetic in the upper grades. Many of the effective materials needed for finding percentages may be classified as visual. The objective materials for teaching the topic are:

a. A hundred board

b. A flannel board with 100 squares or disks, each representing 1 per cent, which will adhere to the flannel board

c. A per cent chart similar to the fraction chart on page 100. The chart can also show the equivalence of certain fractions and their corresponding per cents.

5. *Materials for Teaching Informal Geometry*

The three phases of informal geometry are the geometry of size, shape, and position. The classroom should be equipped with certain manipulative materials for teaching each aspect of informal geometry.

The geometry of size involves the area of plane figures and volume of solids. Effective materials for dealing with geometric figures and finding their dimensions, areas, or volumes include the following:

a. General materials for the classroom

(1) Blackboard compasses and blackboard protractor

(2) A T-square, draftsman's triangle, and drawing board

(3) A carpenter's rule

(4) Eyelet punch with #2 and #3 eyelets

b. Materials for finding dimensions and areas of figures

(1) A square, rectangle, parallelogram, trapezoid, and triangles made of $\frac{3}{16}$ inch plywood

(2) A circular disk 7 inches in diameter

(3) A circular disk about 10 inches in diameter cut into 12 or 16 equal sectors

(4) A circle inscribed in a 20-inch square, ruled in square inches

(5) A 1-foot square ruled to show square inches

c. Materials for finding volumes of solids

(1) Models of a cube, a rectangular prism, a cylinder, a cone, and pyramids having triangular and square bases

(2) A set of 144 1-inch cubes

(3) A cylinder and a cone having equal diameters and equal altitudes

(4) A prism and a pyramid having equal altitudes and congruent bases

(5) Model of a cubic yard

(6) Liquid and dry quart measures and a liter

(7) A metal cubic container having an edge of 1 foot

(8) Cans of different standard sizes as used in storing foods and juices

(9) Specimens of crates or boxes used for packing fruits or vegetables

(10) A slide rule constructed for teaching signed numbers. (See page 422.)

The circular disk having a diameter of 7 inches would be used in deriving the formula for the circumference of a circle. The 10-inch circle cut into equal sections would be used in deriving the formula for the area of a circle. The 20-inch square containing the inscribed circle also would be used in deriving the formula, $A = \pi r^2$. This square with the inscribed circle can be made of heavy cardboard, plywood, or artist's mounting board.

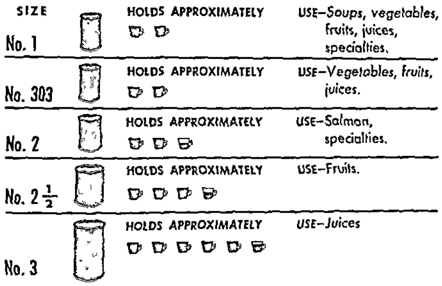
A wooden frame can be used to enclose a square yard. The frame should enclose 9 boards a foot square which should be removable from the square frame. One of the small squares should be divided into 144 square inches.

The models of different solids may be made of wood, cardboard, or plastics. Use at least 125 1-inch cubes of the 144 previously suggested in order to show how to find the volume of a prism. The student can use these cubes to make cubes having 1, 2, 3, 4, or 5 blocks on an edge. He should discover that the number of cubic inches in the volume of a cube is equal to the cube of the number of 1-inch cubes on an edge from his experimentation with cubes having an edge of 5 inches or less. It should not be necessary to use more than 125 1-inch cubes to discover the rule for finding the volume of any cube.

It is important for the student to understand the difference between a liquid quart and a dry quart. In this country a gallon is a measure for liquids only. The imperial gallon, as used in Canada, holds 10 pounds of distilled water. The imperial gallon is defined in both weight and volume, hence it is a measure for both liquid and dry measures. Since a quart of liquid is a quarter of a gallon and a quart of dry measure is $\frac{1}{32}$ of a bushel, a liquid quart is different from a dry quart. There are 4 pecks in a bushel and 8 quarts in a peck, making 32 quarts in a bushel. A bushel is not equal in volume to 8 gallons, thus showing that a liquid quart and a dry quart are different in volume.

A *liter* is equal to 1000 cubic centimeters. A liter of distilled water at standard atmospheric pressure weighs a *kilogram*. Therefore, a liter is a measure for both liquids and solids. A kilogram is equal to approximately 2.2 pounds. A quart of water weighs approximately 2.08 pounds, showing that a liter is a little larger than a liquid quart as measured in this country.

The mathematics classroom of a junior high school should contain a few samples of cans of different sizes which have been used as containers for foods and juices, with labels classifying the amount of the contents of each container. The contents can be expressed as ounces or as *fluid ounces*. Since a fluid ounce frequently is used for labeling of contents of canned foods, the student should know the difference between a fluid ounce and an ounce (avoirdupois). An ounce is $\frac{1}{16}$ of a pound, but a fluid ounce is $\frac{1}{16}$ of a pint or $\frac{1}{128}$ of a gallon. A pint of water weighs slightly more than a pound, therefore, a fluid ounce and an ounce are not the same in weight.



Some cans of the most familiar sizes and their numbers are given in the picture.[2] A No. 1 can will hold 16 ounces of solids, such as salmon, but only 15 fluid ounces of juice. Similarly, each of the other cans will hold a different amount of a solid from the amount of a fruit juice. The volume of a can of a given size is the same for both the solid and the juice, but the unit of measure is different. The density of a solid is an important factor in the weight of a given quantity of that solid.

The classroom should contain a sample of each of two familiar cans which have no identifiable number. The smaller of these cans, frequently used in vending machines dispensing juices, has a diameter of $2\frac{1}{8}$ inches, a height of $3\frac{1}{4}$ inches, and a volume of $5\frac{1}{2}$ fluid ounces. The larger of these two cans has a diameter of $4\frac{1}{4}$ inches and a height of $3\frac{1}{4}$ inches. The height of each can is the same, but the diameter of the larger can is twice the diameter of the smaller can. Therefore, the volume of the larger can is four times the volume of the smaller can. The student should compare the volumes of these cans and thus discover how doubling the diameter of a cylinder affects its volume when the altitude remains constant. See page 408 for a discussion of the procedure to follow in using these cans for instructional purposes.

[2] Based on *A Guide to Common Can Sizes.* New York: The American Can Co.

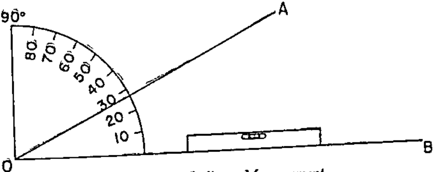
Public Schools, Portland, Oregon

One good use of packing cases is the comparison of a cubic yard with a cubic foot. Foot and yard and square foot and square yard are also seen.

Some teachers of mathematics may object to having in the classroom samples of boxes and packing crates in the list of materials. A few boxes, crates, or containers as used for packing fruits and vegetables may be used in studying volume. There are approximately 200 different types and sizes of crates, boxes, and cartons used for packing fruits and vegetables.[3] Since there are so many different types of containers, the teacher would be able to have in the classroom only a few samples of the most common kinds of containers.

The eyelet punch and eyelets are useful in forming geometric patterns and in curve stitching. Chapter 14 explains how to use these materials.

[3] *Containers for Fruits and Vegetables*, p. 12. Farmers' Bulletin No. 1821. U. S. Department of Agriculture, 1939.



6. *Materials for Teaching Indirect Measurement*

Students need certain instruments for finding inaccessible heights or distances, such as the height of a tree or the width of a stream. Many of the instruments used in indirect measurement may be made by the students. Some of the essential materials used for teaching indirect measurement are a quadrant, a transit, and a surveyor's tape.

A quadrant consists of a protractor attached to a frame with one movable arm, OA, and one fixed arm, OB. The fixed arm contains a spirit level. To find the height of a tree, the student measures a distance easily scaled, such as 100 feet, from the foot of the tree. From that point he sights along the upper edge of the movable arm OA until the point A is in line with the top of the tree. Angle AOB is the angle the tree intercepts from the point O. Page 355 shows how to find the height of the tree when the angle at O and the distance OB are known.

The protractor of a quadrant may be made by scaling to degrees a quarter sector of a circle of convenient size, such as a 10-inch circle. Chapter 14, page 542, gives detailed instructions for making a quadrant.

If a teacher plans to equip the classroom with commercial materials, the trade name for a quadrant would be a *clinometer* or a *hypsometer*.

7. *Miscellaneous Materials*

The teacher may find it possible to use replicas of gas and electric meters to teach the student how to read these meters.

A replica should contain movable dials modeled after the dials on each kind of meter. The dials should be made of wood and attached to a piece of plywood. They should be at least 6 inches in diameter so that the numbers on each dial can be read easily. A fine exercise for the use of a protractor is to have the student mark off 10 points on a circle comparable to the 10 markings on each dial.

The mathematics classroom of the junior high school should contain a demonstration slide rule. Many of the more able students should be taught how to multiply, divide, and find square root by using a slide rule. This is especially valuable for purposes of enriching the curriculum. A demonstration slide rule is very helpful to use for teaching the class how to use a student's slide rule.

c. Visual Materials

Types of Visual Materials

Visual materials include pictures, charts, diagrams, posters, graphs, films, and filmstrips. Visual materials are not so abstract as symbols, but more abstract than objects. Visual materials can be an effective means of bridging the gap between the use of exploratory materials and symbolic materials. The slow learner in particular frequently needs to have this gap between exploratory and symbolic materials bridged.

The use and function of visual aids deserve close examination. The point at issue was well stated by the report from the American Textbook Publishers Institute to its membership.

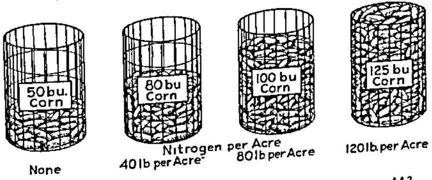
> What about the use of visual aids—pictures, cartoons, diagrams, maps, pictorial graphs, charts? Do they really teach? Are they and the text doing the job together as you would do it if you were talking to your class and had pictures to make clear certain of the ideas that you wanted to give them—or are these graphic materials just "added on" to impress you? Visual materials in a textbook may be like paint on a house, and good-looking paint can be applied to a pretty poor structure.[4]

[4] *Textbooks in Education*, pp. 89–90. New York: The American Textbook Publishers Institute, 1949.

Most of the pictures in mathematics textbooks for the upper grades may be classified as *associative* or *decorative pictures*. If a unit deals with baseball, a picture of an activity associated with baseball sometimes is used to introduce the unit. The picture does not help in the solution of the problems dealing with the unit, but the picture humanizes the book and makes the page more interesting than a solid page of problems. Such a picture informs the student about the theme of the unit.

A picture which supplies data that are used in solving problems dealing with the topic portrayed may be classified as a *functional picture*. The picture below shows: The number of bushels of corn per acre is affected by the use of nitrogen applied to the soil. The student must use the picture to find the data that are used in the problems on the same page. The need for functional pictures in a textbook is not so great in the upper grades as in the primary or intermediate grades. Most textbooks in arithmetic for the upper grades provide a very limited number of functional pictures. If a picture is functional, the student must study and explore it. On the other hand, the teacher cannot be certain that a student will use a picture which is not functional. A picture on a page is called non-functional if it is of no direct help in solving problems on that page.

The teacher should be certain that the students understand a functional picture before they solve the problems based on the picture. The students should be able to identify the data given. In the picture below, the students should be able to read the



yield of corn per acre without the use of nitrogen and then the yield when a certain amount of nitrogen is added to the soil. He should discover that adding 80 pounds of nitrogen doubles the yield of corn. Similarly, other relationships should be discovered before the student is assigned the verbal problems based on the picture.

A diagram is a visual representation of a process. This representation may be either an *illustration* or a *visualization*. Pages 216–217 discuss the difference between these two forms of representation. If a textbook does not visualize a process, the teacher should visualize for the class those processes which can be visualized by diagrams.

The Chalkboard

A chalkboard is the most widely used visual aid in the arithmetic classroom. If the board is used effectively, it is a very valuable piece of equipment in the classroom. One section of the chalkboard should be cross ruled into approximately 2-inch squares. A board of this kind is effective for drawing graphs and for finding the areas of rectangles and other plane figures.

The function of a chalkboard is to show material used for class demonstration purposes. A demonstration on the chalkboard with symbols or diagrams corresponds to the teacher-demonstration with cut-outs on a flannel board. The selection of either of these visual aids depends upon the level of abstraction required by their use. The teacher may have some member of the class volunteer to perform the demonstrations.

Work assigned a student for the benefit of the class should be done at the board. Sometimes the teacher has the student work at the board in order to make a diagnosis of the work. When the chalkboard work is done for the benefit of the class, or a group within the class, the following rules pertaining to the work should be observed:

1. *The board should be cleaned before new work is put on it.* Frequently, a teacher or student who wishes to use the board will find it covered with work. Sometimes posters cover part of the board. The work should be erased and the posters should be

removed so that the student's attention is not disturbed or diverted from the main work.

When a classroom is shared, the opportunity to use the chalkboard frequently is curtailed. The other teacher may have copied on the board some important material to which reference must be made for several days. The sign on the right is familiar to most teachers who share the same classroom. If the teacher of mathematics has need of a chalkboard, it may be necessary to compete with the distraction caused by the work of a colleague. Some arrangement should be made with that instructor.

DO NOT ERASE

2. *The work on the board should be neat.* A student learns to make his work neat by copying the style of work which he sees on the chalkboard, especially the work demonstrated by the teacher. Figures should be legible and written large enough to be read by the students who are farthest from the board. Columns of figures should be written in vertical and not in slanting columns. The sequence of steps in the solution of a problem or example should be identified easily.

3. *Geometric figures and diagrams should be drawn with sufficient accuracy for the student to discover the correct relationship portrayed.* The use of protractor and straightedge is recommended for drawing most geometric figures, especially during initial instruction.

4. *A student assigned a place at the chalkboard for work should have ample space for his work.* He should have one section, approximately 30" × 42", for his work. The work should be labeled properly as discussed on page 327

5. *The teacher must watch the board for glare caused by the sun's rays.* The teacher must view the board from different parts of the room to detect glare. Usually glare can be controlled by curtains in the windows. Because of the difficulty of reading a test on a chalkboard, all tests except those from a textbook should be dittoed, mimeographed, or printed.

Projection Materials

Any material which can be projected on a screen may be classified as projection material. Films, slides, and filmstrips are

the most familiar kinds of projection materials. The number of films and filmstrips which are adaptable for the upper grades is growing rapidly.

A film or a filmstrip may deal with a topic which is predominantly informational, as the *Story of Weights and Measures*, or with the formulation of mathematical concepts, as *What Are Fractions?* Projective material is more useful when dealing with the informational phase of a topic than with the development of a mathematical concept. In the latter case, the film may be a means of presenting the activities the class should do to learn the process or the topic. Few students will understand processes with fractions by observing a film dealing with the topic. If the student has had experiences similar to the scenes depicted on the screen, he usually will learn the topic meaningfully. A teacher will discover from a film or a filmstrip the kind of activities to use to present a topic which is predominantly mathematical and not merely informational. Grossnickle and Metzner summarized the functions of films and filmstrips to be as follows:

1. To motivate the study of number through showing a lifelike usage of number in a meaningful situation.
2. To show the teacher the type of procedure to use or to follow in presenting a particular phase of a topic or process.
3. To introduce activities that provide or suggest means of manipulating materials so as to make learning effective for the pupil.
4. To present an over-all view of a particular topic that may be used as a review or as a form of test.
5. To be used as a means for enriching the professional preparation of teachers. This may be done with students in professional schools or with teachers in the field.[5]

The use of films and filmstrips should be effective in the preparation of teachers of arithmetic. A visual representation can show the prospective teacher the kinds of materials to use and how to use them for teaching a process. The use of television should be extremely valuable for showing how a skillful teacher would present a topic and how to provide learning situations in

[5] Grossnickle, Foster E. and Metzner, William. *The Use of Visual Aids in the Teaching of Arithmetic*, p. 16. Brooklyn: Rambler Press, 1950.

the classroom. The teacher should understand that these instructional aids are only part of a program for the teaching of mathematics and a supplementary part at that. They can never replace the actual handling of exploratory materials.

Equipment for Projection Materials

To be fully equipped for the use of projection materials, the following things should be available:

1. Motion picture projector
2. Motion picture screen
3. Opaque projector
4. Projector for filmstrips and slides.

The cost of equipping each classroom with the above materials is prohibitive. On the other hand, most schools have some or all of the equipment available. These materials should be accessible for the teacher of mathematics. Schedules should be made in advance of the days these materials are to be used in one of the classrooms or in a room set aside for special visual aid equipment.

The cost of equipping a classroom for films is much greater than for filmstrips. Not only do projectors cost more for films than for filmstrips, but also the films cost many times as much as filmstrips. The cost of a filmstrip in black and white is seldom more than four dollars, but a standard classroom film in black and white costs about $50 and twice that amount if desired in color. Most schools cannot afford to own more than a very limited number of films. In a large city system, the films usually are requisitioned from the central school film library or on a rental basis from commercial distributors.

In addition to the lower cost, there is an educational advantage in favor of filmstrips. Each frame of a filmstrip can be discussed as it is projected. If the continuity of a film is broken repeatedly, the effectiveness of the production is greatly lessened. Many teachers do not have the technical skill and knowledge for operating a motion picture projector, but most teachers can operate a projector for filmstrips. From the standpoint of cost, teaching value, and ease of operation, filmstrips have a wider usage than films in the teaching of mathematics.

Films and Filmstrips for Junior High Schools

There are many films and filmstrips in arithmetic which are adaptable for the elementary and intermediate grades. The number of these projection materials which have been designed specifically for the upper grades is not very great. The teacher who may be interested in films and filmstrips for the level below the junior high school may consult the list given on pages 177 to 185 of *The Teaching of Arithmetic*, The Fiftieth Yearbook of the National Society for the Study of Education, Part II.

The list of projection materials which follows includes films and filmstrips which the authors consider helpful for teaching new topics in arithmetic at the level of the junior high school. The films and filmstrips are listed according to publishers.

Coronet Films

Coronet Building, Chicago 1, Illinois

1. *Measurement*
   1 reel, 16 mm sound film. Collaborator: Harold P. Fawcett. Recommended grades, 5–9.
2. *The Story of Weights and Measures*
   1 reel, 16 mm sound film. Collaborator: Foster E. Grossnickle. Recommended grades, 5–9.
3. *The Language of Graphs*
   1 reel, 16 mm sound film. Collaborator: H. C. Christofferson. Recommended grades, 7–12.
4. *Principles of Scale Drawing*
   1 reel, 16 mm sound film. Collaborator: Harold P. Fawcett. Recommended grades, 7–12.
5. *What Is Money?*
   1 reel, 16 mm film. Collaborator: Paul L. Salsgiver. Recommended grades, 6–10.
6. *Fred Meets a Bank*
   1 reel, 16 mm sound film. Collaborators: I. O. Foster and F. G. Neel. Recommended grades, 6–10.
7. *Instalment Buying*
   1 reel, 16 mm sound film. Collaborator: Albert Haring. Recommended grades, 8–12.
8. *Sharing Economic Risks*
   1 reel, 16 mm sound film. Collaborator: Paul L. Salsgiver. Recommended grades, 8–12.

Encyclopaedia Britannica Films

Wilmette, Illinois

9. *Property Taxation*

 1 reel, 16 mm sound film. Collaborator: H. F. Alderfer. Recommended grades, 8–12.

Johnson Hunt Productions

1133 N. Highland Ave, Hollywood 38, Cal

10. *Percentage*

 1 reel, 16 mm sound film For Cases I and II in percentage. Recommended grades, 7–9.

Young America Films, Inc.

18 E. 41st St., New York City 17

11. *The Meaning of Percentage*

 1 reel, 16 mm sound film. Advisors: William A Brownell and Laura K. Eads. Recommended grades, 7–9.

12. *History of Linear Measures*

 35 mm filmstrip. Educational advisor Henry W Syer Recommended grades, 6–8.

Eye Gate House, Inc

330 W. 42nd St., New York City 18

13. *Decimals and Per Cent*

 This is a series of nine 35 mm filmstrips dealing with decimals and per cents. These filmstrips are adapted for the junior high school.

Institute of Life Insurance

488 Madison Ave, New York City 22

14. *How Life Insurance Operates*

 35 mm filmstrip. 41 frames Recommended grades, 8–12.

15. *How Life Insurance Policies Work*

 35 mm filmstrip. 42 frames. Recommended grades, 8–12.

16. *Planning Family Life Insurance*

 35 mm filmstrip. 46 frames. Recommended grades, 8–12.

Society for Visual Education, Inc.

1345 W. Diversey St., Chicago, Illinois

17. *Using and Understanding Numbers—Decimals and Measurements*

 Seven 35 mm filmstrips in color.

The reader may be surprised by the small number of films and filmstrips mentioned in the above list. The limited number is due to two factors. First, only those projection materials are listed which are designed primarily for teaching new topics at the upper level of the elementary school. Second, only those materials are listed which in the judgment of the writers can be used effectively in the mathematics classroom. The reader should consult the list referred to on page 118, and also "Aids to Teaching," as given in issues of *The Mathematics Teacher*, beginning with February, 1948.

d. Symbolic Materials

The Role of the Textbook

Textbooks and workbooks in mathematics contain the most widely used symbolic materials in this subject. The textbook is a necessity if the student is to learn the subject systematically and effectively. Some teachers try to demonstrate their "progressiveness" by disregarding the textbook. The careful step-by-step development of different topics in most mathematics textbooks today should enable the student to master a process much more easily than when the teacher presents the subject in a random sequence without the use of a textbook. The modern textbook is a "learner's" book.

Most students in the public schools are provided free textbooks in the elementary schools. The American Textbook Publishers Institute reports that ". . . thirty-six states or their political subdivisions and the District of Columbia provide free textbooks for the elementary grades. Eleven other states provide for some free textbooks. Only two states make no provision for free textbooks."[6] When the student must buy his books, there may be delay at the beginning of a term before each student has a textbook. Since most states furnish free textbooks in the first eight grades, there should be few cases in which students do not have the use of a textbook during the entire year's work.

[6] *Op. cit.*, p. 80.

Except in a state in which one basic textbook or several designated textbooks in arithmetic are used throughout the state, committees composed of teachers, principals, or other administrative officers usually recommend the adoption of a textbook for a local school district. The basic consideration in evaluating an arithmetic series should be as follows: To what extent does a series of textbooks make possible, or fit into, a complete program in arithmetic? On page 96 it was pointed out that an effective program requires the use of exploratory, visual, and symbolic materials. The textbook should be geared to the use of exploratory and visual materials. It should contain the necessary visualization of new steps and adequate explanation of each new process to be learned. The textbook should contain the practice material needed to develop skill with operations. It should also contain an abundance of material dealing with the applications of number operations.

Items that should be considered by the teacher or by committees concerned with the evaluation of textbooks are given below. The committee on evaluation may decide to assign a certain number of credits to the items given in order to form a scorecard. It is a common practice to distribute a total of 1000 points among the items rated on a scorecard.

I. Method of Presentation
   1. Step by step development of each operation
   2. Stress placed on the meaning of number and place value and of number operation
   3. Provision for use of exploratory materials to give meanings
   4. Visualization used to extend meanings of processes
   5. Provision for student discovery of relationships and generalization to aid learning and assure retention
   6. Wealth of practice material provided at presentation of new work
   7. Provision of supplementary or optional material available for the teacher

8. Systematic and frequent planned reviews of all new steps and processes throughout the textbook in order to maintain skills
9. Diagnostic tests at frequent intervals to locate and correct weak spots
10. Tests for measuring mastery at the end of blocks of new work
11. Application of processes in social situations
12. Provision for different levels of achievement in different processes.

II. Grade Placement of Subject Matter
   1. Gradation of topics based on scientific research and on recommendation of yearbooks and leading courses of study
   2. Spiral arrangement of contents at each grade level
   3. Content of difficult topics spaced through several grades and not bunched in one grade
   4. Flexibility of content possible through omission of optional sections as indicated in a guide for the teacher.

III. Problems
   1. Problems related to a central theme or topic and not a series of unrelated questions
   2. Miscellaneous problems predominantly of the type that can be solved without use of pencil
   3. Content of problems drawn from various curriculum areas, such as social studies, health, science, and arts
   4. Problems based on information in tables, graphs, charts, and other visual materials to develop ability to interpret them
   5. Emphasis placed on the quantitative thinking involved in formulating and applying relationships and generalizations
   6. Provision for use of community resources to vitalize the learning experience
   7. Use of exploratory materials, dramatization, and visual aids to derive solutions of problems.

IV. Provision for Individual Differences
   1. Comprehensive diagnostic program geared to remedial work
   2. Provision of supplementary practice for students requiring more drill
   3. Enrichment exercises and special work for students not needing remedial work
   4. Topics for independent research and reports
   5. Workbook providing special types of aids for slow learners, including exploratory and visual experiences.

V. Testing Program and Provision for Related Remedial Measures
   1. Diagnostic review tests at beginning of year's work keyed to self-help remedial work
   2. Developmental diagnostic tests at intervals in the development of each new operation in whole numbers, fractions, decimals, and per cent
      (a) At least three samples of each step difficulty
      (b) Tests are keyed to study helps and to remedial exercises
   3. Achievement tests covering each block of new work
   4. Cumulative progress tests including all major processes previously taught and keyed to remedial work
   5. Cumulative progress tests in problem solving
   6. Systematic review and test of vocabulary of new concepts introduced.

VI. Authorship
   1. Professional contributions to the field of arithmetic
   2. Success previously demonstrated in preparing textbooks and other instructional materials.

VII. Physical Features
   1. Durability of books
   2. Quality of paper
   3. Sufficient margins.

VIII. Extra Teaching Features
   1. Useful index

2. Use of pictures that are functional and diagrams of various kinds that are visual aids to teaching.

IX. Important Supplementary Considerations
   1. Adequacy of aids given in a teacher's guide accompanying textbook
   2. Supplementary materials for slow learners
   3. Provision by publishers or authors for making available various types of exploratory and visual aids to facilitate learning
   4. Professional books by authors containing a comprehensive presentation of underlying principles and procedures for teaching arithmetic.

The Mathematics Workbook

A workbook should have a specific function to perform in a mathematics program. The workbook should not be merely a drill book or a means of supplying additional examples and problems similar to those given in the textbook. Unfortunately, most workbooks in elementary mathematics do have this limited function. The conventional workbook simply provides more practice in the same activities that the textbook contains. If all members of a class have these drill books, the use of the workbook provides much busy work for many of the more able students because often they have no need for further drill in a given topic or process. The student who does not understand the work in the textbook, moreover, receives little help from such a workbook.

The teacher may use a workbook to provide examples for practice instead of having the student copy them from the textbook. The examples in the workbook should be so arranged that they can be scored quickly and easily. In this case a workbook saves the time of both teacher and student. Such a usage of a workbook is defensible in a program for teaching arithmetic.

Workbooks in mathematics should be written for two groups, namely, the slow learners and the fast learners. There should be a different workbook for each of these two groups. The book for

the slow learner should be written at a different level of abstraction than the book for the fast learner.

The workbook for the slow learner should contain much more visual material than is provided in the textbook. Often a student does not understand a textbook development because of reading difficulties or because of his limited experience with the topic. For example, a student in the upper grades may not be able to add unlike fractions. He needs meaningful experiences with exploratory and visual materials. Most of his classmates may understand the topic, hence they can proceed at once to practice with symbolic materials as provided in the textbook. A workbook for slow learners in the upper grades should provide the development of a topic at a much lower level of abstraction than the development given in the textbook for that grade so as to provide for individual needs of slow learners.

Only those students who have difficulty in understanding the textbook need the detailed guidance a workbook can provide. In a class of 30 students, there may be a need of a workbook for approximately a third or a half of the group. The other students may be able to make satisfactory progress with the textbook.

There may be some within the group making satisfactory progress who need to be challenged with more difficult problems and examples and a wider variety of types than those given in the textbook. This group of fast learners should be given a different workbook than that used by the slow learners. The workbook for the fast learners should contain a minimum of visual materials. The material should include tabular data, graphs, and diagrams as well as verbal abstract materials. The questions asked about a process should be of the type to lead the student to make generalizations which only the fast learners would discover. The problems should be of the type that will challenge the superior student in mathematics.

From the above discussion, it is seen that a workbook has a specific function in a mathematics program. One type of workbook should enable the slow learner to extend a background in a topic so that he will succeed with the textbook. The second type of workbook should be for those students who need greater enrichment in their quantitative thinking than is given in their

textbook. Because of our grouping and promotional policies, the need of materials for the fast learners is as imperative as the need for materials for slow learners. Chapter 14 gives means of enriching the curriculum for superior students.

A workbook should be a means of providing for individual differences within a class. If a student needs a workbook, the type of book he should be given depends upon his background in a particular topic. Workbooks should be written so as to provide for specific needs of certain students and not for the general needs of all students.

The textbook and workbook typify the chief symbolic materials for the mathematics classroom. These are not the only books the classroom should contain. Chapter 14 shows that a mathematical library provides one of the chief means of enrichment. This library should contain a variety of books dealing with mathematical topics, magazines, and other printed material which will help to enrich a student's background in the subject.

Questions, Problems, and Topics for Discussion

13. Discuss the function of a chalkboard from the standpoint of a visual aid.

14. State how you think a film or filmstrip should be used so as to achieve its greatest educational value to the student in arithmetic.

15. Make a list of films or filmstrips which would be effective for teaching the meaning of common fractions and the basic processes with fractions.

16. If you were a member of a committee to select a textbook in arithmetic, how would you decide on the book to be adopted in your school system?

17. Make an analysis of a workbook in arithmetic and see how it fits into a program for providing for individual differences among the students of a class.

18. Debate the following: Resolved The teacher and students should make most of the exploratory materials used in the mathematics classroom.

Suggested Readings

Bartnick, Lawrence P. *Designing the Mathematics Classroom*, p 44 Washington, D. C.: The National Council of Teachers of Mathematics, 1956

Bedford, Fred L. "Designing the Mathematics Classroom," *The American School and University: Eighteenth Annual Edition*, pp. 199–202. New York· American School Publishing Corporation, 1946

Brueckner, Leo J. and Grossnickle, Foster E. *Making Arithmetic Meaningful*, Chapter II and pp. 533–558. Philadelphia. The John C Winston Co., 1953.

Carroll, L. Grace "A Mathematics Classroom Becomes a Laboratory," *Multi-Sensory Aids in the Teaching of Mathematics*, pp 16–29 Eighteenth Yearbook of the National Council of the Teachers of Mathematics. New York. Bureau of Publications, Teachers College, Columbia University, 1945

Emerging Practices in Mathematics Education, pp. 131–151 Twenty-second Yearbook of the National Council of Teachers of Mathematics Washington, D. C.: The Council, 1954.

Grossnickle, Foster E., Junge, Charlotte, and Metzner, William "Instructional Materials for Teaching Arithmetic," *The Teaching of Arithmetic*, pp 155–185 Fiftieth Yearbook of the National Society for the Study of Education, Part II Chicago: University of Chicago Press, 1951.

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Kidd, Kenneth P. and Brown, Kenneth E. *Teaching Materials for Mathematics Classes*, p. 36, Circular No. 399. Washington, D C.: U. S. Department of Health, Education, and Welfare.

Kinney, Lucien B. and Purdy, C. R. *Teaching Mathematics in the Secondary School*, pp. 290–303. New York: Rinehart & Co., 1952.

Reeve, William D. *Mathematics for the Secondary School*, pp. 486–509 New York: Henry Holt and Co , 1954.

Spitzer, Herbert F. *The Teaching of Arithmetic*, pp. 318–45. Boston: Houghton Mifflin Co., 1954.

Urbancek, J. "Mathematical Teaching Aids," *Chicago Schools Journal*. 35:1–80.

Chapter 5

Developing Power in the Basic Processes with Integers

In the junior high school, the student should deal with integers at a higher level of operation than in the earlier grades. This chapter discusses how to deal with integers in the four basic processes under the following major headings:

a. Elements of a minimum program in the basic processes
b. Meanings in arithmetic
c. Mathematical principles governing the basic processes
d. Diagnosis and treatment of learning difficulties in arithmetic
e. Discovery of relationships among processes.

a. Elements of a Minimum Program in the Basic Processes

Growth in Ability to Deal with Integers in the Basic Processes

The teacher of arithmetic in the upper grades has a twofold task to perform in dealing with integers in the basic processes. First, the teacher must provide learning experiences which will enable students to grow in ability to deal intelligently with numbers. Second, the teacher must adjust the curriculum and methods of instruction to the wide range of ability and rates of learning of the students. All students should master the processes that are undoubtedly of social utility. The students also should acquire an understanding and appreciation of the important

social institutions in which number plays an important role, such as banking, taxation, insurance, and the mathematics of the home.

One of the chief attributes of a program which stresses meanings in arithmetic is its provision for growth in ability to deal with quantities. A student who reviews a process should begin at the level at which he can understand the work. A student who must begin at an immature level of operation should progress through successively higher levels of abstraction until ultimately he can check an answer by approximation to determine whether or not the answer is sensible or reasonable. There is a wide range of achievement between the ability of the student who must use exploratory materials or visual aids, to subtract in example A, as shown on the right, and the ability of the student who subtracts in example B and checks the answer by the following thought pattern: "Since 958 is a little less than 1000, the answer must be a little more than 7700."

A
$$\begin{array}{r} {}^{4}\;{}^{13} \\ \cancel{5}\;\cancel{3} \\ -2\;\,8 \\ \hline 2\;\,5 \end{array}$$

B
$$\begin{array}{r} 8\,7\,2\,6 \\ -\;9\,5\,8 \\ \hline 7\,7\,6\,8 \end{array}$$

In the upper grades some slow learners who do not understand the method of subtracting numbers can be helped by writing out the steps as shown in example A. As a student becomes familiar with the method of regrouping numbers as needed in the algorism and understands the process, he begins to operate without the use of supplementary aids. Then he reviews the method of subtracting two three-place numbers in examples involving more difficult compound subtraction. When he is able to subtract in any example given in his textbook with a reasonable degree of speed and a high degree of accuracy, many teachers assume that he has attained the degree of mastery which is too often accepted as the goal of teaching the process. An achievement of this kind can be attained by drill and repetition in a program which does not emphasize a high level of performance. Mastery of a process is not attained until the student is able to check the reasonableness of an answer by approximation. He must be able to make a quick estimate of approximately what the correct answer should be.

Many students who can work an example in a process cannot determine with assurance whether or not the answer is sensible. They may be able to check the answer to prove that it is correct, but this also may be a mechanical operation that actually is not understood. A student demonstrates mastery of a process when he is able to verify an answer by use of *round numbers*. When he reaches this stage in his thinking with numbers, he deals intelligently with numbers at the highest level of operation which characterizes literacy in the use of number.

Widely Used Checks in the Basic Processes

Checking an answer for accuracy is an integral part of the program for teaching arithmetic. The most widely accepted checks for the four processes with integers are as follows:

1. Addition: Add the numbers in a column in the opposite direction
2. Subtraction: Add remainder to subtrahend
3. Multiplication: Go over the work, or interchange the numbers in the example
4. Division: Go over the work, or multiply divisor and quotient and add the remainder, if any, to that product.

The value of checking, providing the procedure is understood, is not to be minimized. It is imperative, however, for the teacher to understand that mechanical checking for accuracy does not lead to growth in ability to think with numbers. The intelligent use of round numbers in approximation to show if an answer is sensible will contribute to growth in ability to deal with numbers.

Elements of a Minimum Program in Basic Processes

It is not possible to state what should constitute all of the elements of a minimum program in arithmetic and mathematics for the upper grades. However, certain essentials of the subject have such great social utility that their inclusion in a program of this type should not be controversial. The student should master all the basic facts in the four processes, understand the

meanings of each process, and be able to do the algorisms in these processes intelligently and skillfully. The first and second items are definite and it is easy to measure the student's achievement or understanding in these areas. It is hard to determine the degree of difficulty of an example a given student should be able to perform in a process. For example, should he be able to divide by one-, two-, three-, or more-place divisors? It is evident that a minimum program for all students cannot set as its goal the ability to divide by a four-place divisor. The criterion of social utility should be applied rigorously in the selection of material for a desirable program for slow learners in arithmetic. On that basis the slower students should be able to divide by one- or two-place divisors. To meet this level the slow learner should not be expected to master a highly efficient method of estimation of the quotient, but he should understand the meaning of division so that when necessary he can find the answer of an example involving this process. Similarly, the slow learner should be able to show mastery of the other basic processes in dealing with the most common kinds of examples used in daily affairs. For more difficult examples in each process, this student should be able to find the answers but without using the efficient methods demanded of more able students

Finding the Level of Performance of the Student

At the beginning of each school year the teacher should give an *inventory test* to determine how well the students have mastered the work of the previous grades. A test of this kind need have only a limited sampling of the work the student has had. Since the sampling is limited, the results of a test of this kind do not give a very reliable index of a student's mastery of a particular process. However, glaring deficiencies in processes can be detected from an analysis of the results of an inventory test. The inventory test on page 132 is typical of the kind of test to be given early in the first year of the junior high school. The test samples the student's ability to perform the basic processes with integers and common and decimal fractions. At the beginning of the eighth and ninth grades similar tests should be given, but

these tests should include examples which are representative of the work given in per cent during the previous years.

Inventory Test

I. Addition:

1.	2.	3.	4.	5.	6.
584	$ 5.89	$3\frac{1}{2}$	$2\frac{7}{12}$	.96	74.5
967	23.17	$9\frac{5}{8}$	$6\frac{1}{3}$	.75	9.6
358	1.98	$6\frac{1}{4}$	$9\frac{3}{4}$	.48	21.8
895	46.50	$5\frac{7}{8}$	$4\frac{1}{2}$	.35	3.4
608	8.49				

II. Subtraction:

1.	2.	3.	4.	5.	6.
5243	70,325	$12\frac{1}{2}$	$16\frac{1}{2}$	.534	7.25
3645	9,328	$7\frac{3}{4}$	$8\frac{2}{3}$	.258	4.56

III. Multiplication:

1.	2.	5.	6.
97	708	4.85	7.25
68	407	8	1.4

3. $2\frac{1}{2} \times 5\frac{1}{3} =$

4. $8 \times 6\frac{2}{3} =$

IV. Division:

1. $34\overline{)70,385}$

2. $27\overline{)9561}$

3. $72 \div \frac{8}{9} =$

4. $2\frac{1}{3} \div 5\frac{1}{4} =$

5. $25\overline{)1.6}$

6. $.8\overline{)23}$

Next, the teacher should give *diagnostic tests* in each process to determine the student's weak spots and deficiencies in the four basic operations with integers and common and decimal fractions. A diagnostic test will show in what elements and at what level of difficulty a student has trouble with a process. (See Chapter 13.)

The results of the illustrative diagnostic test on page 133 will show whether or not the student is able to work all types of examples in division of integers when the divisor is a two-place

number. All examples in each row are similar in structure. If a student should make an error in only one example in a group and have all of the remaining examples in that group correct, the likelihood is that the error is due to chance. On the other hand, if answers to two or more examples in a group are incorrect, the errors indicate that very probably the student does not understand how to work that type of example. Apparently he needs corrective help on that particular phase of the process.

Diagnostic Test in Division by a Two-place Divisor

	a	b	c
I.	24)768	36)1483	74)39,628
II.	36)7347	45)15,750	53)18,574
III.	64)2465	35)12,668	96)36,920
IV.	67)3241	46)40,235	87)58,793
V.	26)18,363	38)28,421	16)9123

All of the examples in each row of the above test represent the same structural difficulty. The examples in each row may be classified as follows:

I. The estimated quotient is the true quotient
II. All zero types of the quotient are given
III. The true quotient is one less than the estimated quotient when the need for correction is evident
IV. Same as III but the need for correction is not evident
V. The true quotient is two or more removed from the estimated quotient.

The examples in each group are classified according to the type of estimation of the quotient figure. Errors in division may result from many other sources besides those made in estimation of the quotient figure. Frequently the teacher must use more

analytical methods of diagnosis (see Chapter 13) to determine a student's difficulty in a given type of example in division.

The diagnostic test on page 133 typifies the kind of test that should be given in each of the processes with integers and common and decimal fractions. A well constructed diagnostic test[1] of this kind enables the teacher to determine at what point the student needs a review of previous work. In some cases it may be necessary practically to reteach a given process. If a diagnostic test shows that a particular student does not know some of the basic number facts in a given process, the teacher should use at the junior high school level the same procedures that are the accepted procedures for teaching these facts in the lower grades. This also applies to the reteaching of number operations in disability cases.

Testing the Student's Knowledge of the Basic Facts in Addition

A *basic fact* in a process is the grouping of any two one-place numbers with the answer. Thus, $3 + 9 = 12$ is a basic fact in addition and $6\overline{)42}$ with 7 as the answer is a basic fact in division. The indicated operation of two one-place numbers, as $4 + 5$, is a number *grouping*[2] in addition. Including the zero facts, there are 100 basic facts in addition and in subtraction. Zero is not used as a multiplier or as a divisor, hence there are only 90 basic facts in multiplication and in division. The teacher should give a systematic test of the facts in each process to determine the facts the student does not know. Usually, students in the upper grades will know the basic facts in addition which have a sum less than 10 and the corresponding facts in subtraction. These students also will know the facts involving the 2's, 3's, 4's, and 5's in multiplication and the corresponding facts in division. The other facts frequently require careful restudying.

[1] The reader will find a complete series of such diagnostic tests in *Winston Arithmetics.*

[2] The expression "grouping" is used to identify expressions without answers, such as $2 + 1 =$, $4 - 3 =$, $3 \times 4 =$, and $12 \div 6 =$. The expression appears in Rosenquist, Lucy *Young Children Learn to Use Arithmetic*, p 23 Boston. Ginn and Company, 1949.

Two plans for testing a student's knowledge of the basic facts in a process are given in Chapter 13. The teacher should follow either of these plans to find the facts which the student does not know.

The way the teacher guides the student in his study of the facts not known determines to a large extent whether or not he will learn these facts meaningfully. He may learn them by rote or by applying generalizations. The latter form is a characteristic of meaningful learning. If a student does not know the fact, $9 + 7 = 16$, he can relate this fact to the grouping $10 + 7$, to the grouping $9 + 9$, or to the grouping $7 + 7$. He may be able to discover the relationship between the groupings $9 + 7$ and $8 + 8$. In this way the meaning of the fact, $9 + 7 = 16$ is enriched. This enrichment is not possible when the fact is learned in isolation from the other facts. If the student is unable to deal with symbols, he should use objective materials such as markers to find the sum. Although the work described is characteristic of that ordinarily taught in the third or fourth grade, the teacher must remember that the student may not know certain basic facts in a process. This unit of subject matter should be taught meaningfully for mastery regardless of the student's grouping or classification in school.

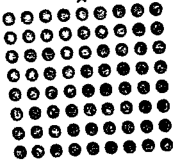
After an interval of a week, the teacher should repeat the test of the basic facts. Then at varying intervals as seems advisable the teacher should repeat the procedure for testing the student's knowledge of the basic facts.

b. Meanings in Arithmetic

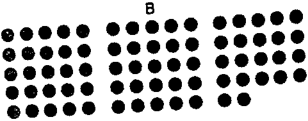
Explaining the Meaning of Arithmetic

It is possible to group objects or things. Arithmetic deals with numbers which represent ways in which objects can be grouped. Two or more groups may be combined into one group, or one group may be separated into two or more groups. The total of two or more unequal groups may be found by counting or by addition; the total of two or more equal groups may be found by counting, addition, or multiplication. One group may be

A



B



separated into two unequal groups by subtraction; one group may be separated into two or more equal groups by either subtraction or division. Division also may be used to find the size of equal groups. Multiplication is a quick way of combining groups that are equal and division is a quick means of separating one group into a number of equal groups or finding the size of each group formed. Two groups are compared by either subtraction or division.

Counting by ones is a satisfactory procedure to use to find the number of items in a group of not more than 10 things. Counting or adding or multiplying is a possible way of finding the number in two or more groups of more than 10 things each. Risden[3] showed that many pupils in the elementary school are unable to group objects or numbers to find the total amount. The value of grouping as a means of dealing with a collection of things can be shown in the diagram. In diagram A, the circles are so arranged that it is difficult to group them to find the number in either a row or a column, hence counting by ones is a satisfactory means to use. Each row is treated as a group and the number in a row is multiplied by the number of rows. In this case both counting and grouping are used to find the number of circles in the diagram.

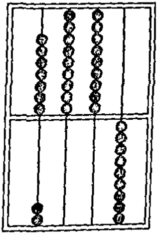
In diagram B, the circles are so arranged that counting by ones is not necessary. The reader discovers that there are 25

[3] Risden, Gladys "What is Wrong with School Arithmetic?" *The Mathematics Teacher*, 46:407-11.

circles in each of the first two groups and 3 circles less than 25 circles in the third group, making a total of $25 + 25 + 25 - 3$, or 72. It is possible also to make groups of $25 + 25 + 20 + 2$ for a total of 72. The circles in B can be arranged into many other groupings which the reader could recognize easily to enable him to find the total number of circles much more quickly than by counting by ones. Groups of 2, 3, 4, or 5 items are easily identified in a visual representation of a quantity, such as given on the opposite page. The ease of determining the number of items in a group depends upon the form of grouping employed.

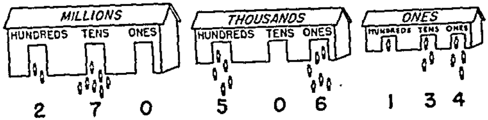
The Structure of Our Number System

In order to understand how to deal with groups and regrouping, it is necessary to understand the structure of our number system.[4] The distinguishing feature of our number system is *place value* which is made possible by the use of zero as a place holder. Without zero, place value cannot be shown except by some label or mechanical device, such as an *abacus* having nine beads on a rod. Each bead represents one of the digits from 1 to 9. The number shown on the abacus is 2009. The absence of a bead on a rod is shown by zero in the written number.



Our number system is *decimal* in form, which means that the system of notation has a base of ten. In this system the value of a figure one place to the left has ten times its value one place to the right. Similarly, moving a figure one or more places to the left multiplies the value of the figure by a power of ten for each place it is moved. Inversely, moving a figure one or more places to the right divides its value by ten or a power of ten for each place it is moved. At one thousand and beyond, numbers

[4] For a full discussion of this topic, see Brueckner, L. J. and Grossnickle, F. E. *Making Arithmetic Meaningful*, Chapter 2. Philadelphia: The John C. Winston Company, 1953.



are grouped into periods or families of three digits, and each period is given an identifying name, as shown in the diagram above.

The period to the left of millions is *billions*. Thus, there are 1000 millions in a billion. The manikins in the diagram show how the nine digits may be used in each place in a group. Zero is used to show an empty place in a group.

Scientific Notation

The principle of place value makes it possible to express a number in *scientific notation*. Frequently large numbers, as used in astronomy, are written in scientific notation. The number, 4,500,000, may be written in this notation as 4.5×10^6. To express a number in scientific notation, divide the number by 10 raised to the power that will give a quotient having a value of at least 1 but less than 10. In a whole number, the power of 10 is the same as the number of places in the given number to the left of ones' place. Thus, 375,000,000 should be divided by some power of 10 to give a quotient of 3.75. (The original number must be divided by 10^8 to give a quotient of 3.75.) To keep the value of the given number the same, 3.75 must be multiplied by 10^8; hence 375,000,000 may be expressed as 3.75×10^8.

Similarly, a decimal fraction may be presented in scientific notation by use of a negative exponent. The number, .000025, may be written as 2.5×10^{-5}. The given decimal must be multiplied by 10^5 to have a value of 2.5. To keep the value of the decimal the same, 2.5 must be divided by 10^5 which is the equivalent of multiplying by 10^{-5}. The number is therefore written as 2.5×10^{-5}.

Grouped, Ungrouped, and Regrouped Numbers

Each figure in a number has a *grouped value* and an *ungrouped value.* The grouped value of the 6 in 60 is 6 tens. The ungrouped value of the 6 in 60 is 60 ones. The grouped value of a figure in a number is the absolute value of that figure. Each grouped number shows the frequency of its base. In the number 347, the grouped values of the three digits are, respectively, 3 hundreds, 4 tens, and 7 ones. The ungrouped value of these digits depends upon the amount of transformation or grouping made. Thus, the number 347 is the same as 3 hundreds and 47 ones, 34 tens and 7 ones, or 347 ones.

Most teachers emphasize identification of the grouped or absolute value of the digits in a two- or more-place number. Often these teachers neglect to instruct the student how to determine the ungrouped value of a number. It is easy to teach placement of the quotient meaningfully when a pupil is able to give the ungrouped value of the digits of a number. In the example, $36\overline{)110}$, the pupil should see that it is not possible to divide 11 tens into 36 equal groups of tens; then it is necessary to change 11 tens to 110 ones and to divide 110 ones into 36 equal parts. Since 110 ones are being divided, the 3 is written in the ones' place in the quotient.

Not only must the pupil understand both the grouped and the ungrouped values of the digits of a number, but also he must understand how a number may be *regrouped.* To subtract in Example A, it is necessary to regroup 64 as 5 tens and 14 ones. Many teachers speak of this transformation as "borrowing a ten." This phrase is not descriptive of the operation. To subtract in Example B, two successive regroupings are needed. The reader should be able to state how 724 must be regrouped.

A
$$\begin{array}{rr} {}^{5} & {}^{14} \\ \cancel{6} & \cancel{4} \\ -2 & 8 \\ \hline \end{array}$$

B
$$\begin{array}{r} 7\;2\;4 \\ -1\;5\;8 \\ \hline \end{array}$$

The need for regrouping can be illustrated clearly in division of decimals. In order to change the common fraction $\frac{1}{2}$ to a decimal fraction, it is necessary to regroup the 1 of the numerator as 10 tenths. Then it is possible to divide the numerator as shown in the example, $2\overline{)1.0}$, and the quotient is 5 tenths, or .5.

It is almost impossible for a pupil to understand this process unless he uses objective and visual materials, as is shown in Chapter 6, before he deals with such symbolic representations.

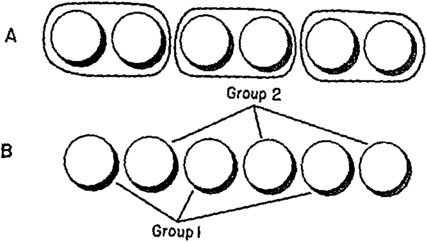
From the illustrations given, it is evident that a student must learn how to deal with grouped, ungrouped, and regrouped numbers. In the addition example shown, the sum of the numbers in ones' place is 16 ones, an ungrouped number. This number is grouped as 1 ten and 6 ones and the 6 ones are written in the ones' column in the sum. The 1 ten is "carried" to tens' place. Then the number of tens is added. If we were to subtract in an example, such as 28 from 76, it would be necessary to regroup 76 as 6 tens and 16 ones.

$$\begin{array}{r} 4\ 7 \\ +2\ 9 \\ \hline 7\ 6 \end{array}$$

Meaning of the Processes

According to the plan on pages 130–131, the minimum achievement with integers in the upper grades should be: (1) a knowledge of all basic facts; (2) an understanding of the fundamental operations; and (3) the ability to perform the algorisms in the four processes. We have discussed how to test for knowledge of the basic facts and how to reteach the facts not known.

The second part of the minimum program dealing with integers should have as its goal the development of an understanding of the four basic operations. It is unlikely that at the junior high school level there will be students who do not know that addition is the process of forming one group from two or more groups, or parts of groups as in fractions, or that subtraction is the process of separating one group into two groups. If the student understands the basic facts in multiplication, he must have discovered the relationship between addition and multiplication. He should know that a multiplication fact, such as $4 \times 7 = 28$, can be found by adding 4 sevens or 7 fours. The student at the junior high school level usually understands the meaning of addition, subtraction, and multiplication. Many students at this level do not understand the two meanings of division.



Division shows (1) how many times as large one number is as another number and (2) the size of the equal parts into which a number is divided. Diagrams A and B show these two uses or meanings. In the first usage, in A, known as *measurement*, division is a shortened form of subtraction. The size of the total group is known, but the number of equal groups of a given size that may be formed is not known. In the second usage, in B, known as *partition*, the number of equal groups to be formed is known, but the size of each group is not known and must be found.

If a student does not understand the difference between the two uses of division, he should be shown how to use objective materials to demonstrate each use. If 12 pupils are to be divided into groups of 4 pupils each, this solution can be objectified with 12 markers, such as disks. The student takes away in succession 4 disks from the group of 12 disks until there are no disks left and they are separated into 3 equal groups. In this case 4 is subtracted from 12 three times. If 12 pupils are to be divided into 3 equal groups, this solution can be objectified by sorting 12 disks one at a time into 3 equal piles. Each pile or group will contain 4 disks. The student should use similar objective and visual materials until he understands the difference between the two usages and can state problems to represent each usage.

Mastery of the difference between the two uses of division can be demonstrated by the ability of the student to discover

the relationship among divisor, dividend, and quotient in given examples. In the use involving measurement, the student should discover that both divisor and dividend are expressed in the same unit, but the quotient is an unlabeled number. In the example, $\$6\overline{)\$30}^{\,5}$, divisor and dividend are expressed as dollars, but the quotient is abstract. This form of division is used whenever two numbers are compared, as in finding a ratio or what per cent one number is of another number. In an example involving partition, dividend and quotient are expressed in the same unit, but the divisor is an unlabeled number. In the example, $6\overline{)\$30}^{\,\$5}$, dividend and quotient are expressed as dollars, but the divisor is abstract. Most superior students in the upper grades are able to make generalizations about the correct labeling of the numbers given in a problem as used in a division example of this kind.

c. Mathematical Principles Pertaining to the Basic Processes

An Analysis of Principles

The student in the upper grades should discover some of the basic generalizations which pertain to grouping within each of the four fundamental processes. The following generalizations or principles characterize the operations within the various processes. These principles apply also to processes with algebraic or general number. The most important generalizations are given below.

I. Addition and Subtraction

1. *Only like quantities and values can be added or subtracted.* The pupil soon learns that he cannot add apples and pennies. This principle applies also to numbers in like places. According to this principle, only ones can be added to ones, tens to tens, or tenths to tenths. One cannot add halves and tenths directly until their form has been changed.

2. *Numbers can be added in any order.* This principle can be illustrated in the example on the right. In adding a column, a pupil frequently skips about the column in order to make a combination of 10 or some other combination. This is an inefficient procedure but it gives the correct answer. From illustrations of this kind the student should generalize that the order in which numbers are added does not affect the sum.

$$\begin{array}{r} 8 \\ 4 \\ 2 \\ 6 \\ \underline{7} \end{array}$$

3. *If either of two numbers is subtracted from their sum, the remainder is the other number.* The pupil should discover this principle in dealing with the basic facts in addition and the corresponding facts in subtraction. From the fact, $3 + 4 = 7$, the pupil should discover that $7 - 3 = 4$ and $7 - 4 = 3$, showing that subtraction is the inverse of addition. It is possible to make four examples by using the numbers in the box. These examples illustrate principles No. 2 and 3.

3	4	7

II. Multiplication and Division

1. *The order in which two or more numbers are multiplied does not affect the product.* Most pupils discover this principle when they learn the basic facts in multiplication. The pupil should know that 2×3 has the same product as 3×2. It follows then that $a \times b$ has the same value as $b \times a$.

2. *An indicated sum of two or more numbers can be multiplied or divided by a number providing each term of the indicated sum is multiplied or divided by that number and the sum of the products or quotients is found.* This principle can be illustrated by multiplication or division of integers. Multiplying 37 by 2 is the same as simplifying the expression $2(30 + 7)$, and dividing 64 by 2 is the same as $2\overline{)60 + 4}$. To find the value of $2(30 + 7)$, both 30 and 7 must be multiplied by 2 and the products added. Similarly, to find the value of $2\overline{)60 + 4}$, both numbers must be divided by 2 and the quotients added. The numbers 30 and 7 may be added and this sum multiplied by 2. This plan usually is followed in evaluating a formula. In the formula for finding the perimeter of a rectangle, $p = 2(l + w)$, both l and w can be multiplied by 2, or their

sum multiplied by 2. Thus, if $l = 7$ and $w = 5$, the perimeter can be found by either of the following ways:

$$p = 2(7 + 5);\ 14 + 10 = 24$$
$$p = 2 \times 12, \text{ or } 24.$$

3. *An indicated product of two or more numbers or factors is multiplied by a number when only one factor of that product is multiplied by that number. Similarly, an indicated product is divided by a number when only one factor is divided by that number.* This principle has direct application in evaluating certain formulas, such as, in evaluating the formula for the volume of a cone. The formula for the volume is $V = \frac{\pi r^2 h}{3}$. If $r = 6$ and $h = 4$, these values may be substituted so that the formula becomes $A = \frac{\pi \times 6 \times 6 \times 4}{3}$. According to the first principle of multiplication, the factors may be multiplied in any order. It is possible to multiply 6 by 6 and this product by 4, or 6 by 4 and this product by 6. Then the product, 144, should be multiplied by π, or 3.14. According to the first part of the principle given above, any one of the factors may be multiplied by π, but not more than one of these factors may be multiplied by π. Now the indicated product, $\pi \times 6 \times 6 \times 4$, must be divided by 3. Only one of the 6's should be divided by 3. The value of V is equal to $\pi \times 2 \times 6 \times 4$, or 48π.

4. *Dividing by a number is the same as multiplying by the reciprocal of that number. Similarly, multiplying by a number is the same as dividing by the reciprocal of that number.* One number is the *reciprocal* of another number when the product of the two numbers is 1. Thus, 3 is the reciprocal of $\frac{1}{3}$, and $\frac{3}{4}$ is the reciprocal of $\frac{4}{3}$. If a number is to be divided by 4, the quotient would be the same as multiplying that number by $\frac{1}{4}$. It is possible to divide by a fraction by inverting the fraction and then multiplying since multiplication and division are inverse processes. The inverted form of a fraction is the reciprocal of that fraction.

5. *If the product of two numbers and one of the numbers are given, the other number may be found by dividing the product by the given number.* This principle has wide application in arithmetic but a limited

application in algebra. The student finds the missing number in the example, $3 \times ? = 12$, by dividing 12 by 3. If we consider the equation, $3n = 12$, in which n represents a certain number, 12 is the product of 3 and n; hence we may find the value of n by dividing 12 by 3. The usual way to solve the equation is to apply the division axiom which states that both members of an equation may be divided by the same number without changing the value of the equation.

6. *The value of a fraction is not changed if both numerator and denominator are multiplied or divided by the same number, except zero.* This principle is usually understood by a student in the upper grades, especially if he has learned how to reduce fractions to lower terms or how to express fractions in higher terms. Since every fraction is an indicated division, the principle applies to the division algorism. In this case the principle may be stated as follows: Both divisor and dividend may be multiplied or divided by the same number, except zero, without changing the value of the quotient. The example, $1.5\overline{).75} = 15\overline{)7.5}$, illustrates the application of this principle.

How to Use the Mathematical Principles

The principles listed above for the four processes are not to be taught as a unit or body of subject matter. The teacher should understand each principle and be aware of situations in which one or more of these principles apply. Many teachers are not sensitive to the possibilities which occur in normal classroom experiences for enriching the mathematical thinking of a student. To illustrate, the student may evaluate the formula for the perimeter of a rectangle, as $p = 2(4 + 7)$. The conventional way is to add 4 and 7 and multiply 11 by 2. The teacher also should have the student find the answer by multiplying each term of the indicated sum by 2 and then adding those products. Then he should generalize about the experience. The student understands the principle when he generalizes about several experiences. The skillful teacher is always alert to quantitative situations in which the student with insight should discover the operation of a mathematical principle.

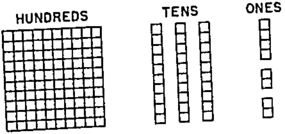
It should be emphasized that the principles discussed above pertain solely to the mathematical phase of number. Most of the slow learners in arithmetic will profit very little from instruction aimed at an understanding of all of these principles. The study of these generalizations dealing with the basic processes is intended primarily for the superior students who will develop insight into number relationships. Students having the ability to form these more difficult generalizations should be encouraged to continue their study in mathematics. If a student understands these principles in arithmetic, he should experience little difficulty in understanding the basic operations with general number in algebra.

d. Diagnosis and Treatment of Learning Difficulties in Arithmetic

An Arithmetic Kit for Slow Learners

Most of the students in the upper grades know how to perform the algorisms with whole numbers in the four basic processes. Those students who do not understand the operations with integers and fractions should use materials to discover the meaning of the procedures. Each student in this group should have an *arithmetic kit*. It should contain a set of squares and rectangular strips for exploring integers and decimal fractions and a set of fractional *cut-outs* (see Chapter 4) for exploring common fractions. With these two sets of materials, the student should be able to discover the meaning of the number system and the meaning of the steps in the basic operations with whole numbers and both common and decimal fractions.

The squares and rectangular strips used for dealing with integers can be cut from a sheet of tagboard or cardboard 15″ × 21″. One side of this sheet of tagboard or cardboard should be cross-ruled into $\frac{3}{4}$-inch squares and the other side should be blank. A ruled sheet 15″ × 21″ will contain 560 $\frac{3}{4}$-inch squares which should be cut so as to make 3 pieces of 100 squares each and 20 strips of 10 squares each. The remaining 60 squares should be cut into pieces containing one, two, and other numbers of squares up to ten squares. The large squares



may be used to represent hundreds, the strips of ten squares to represent tens, and the groups of squares and small strips to represent ones. The number represented in the diagram is 138. The student can use different groupings of the single squares to represent the number of ones. For that reason the strips of squares to represent the ones should not all be cut in individual squares.

The teacher should understand that these manipulative materials are for the few students who do not understand the algorisms with integers. Each student must understand that 10 ones can be regrouped as 1 ten and that 1 hundred can be regrouped as 10 tens. He should be able to apply the principle of regrouping in dealing with numbers in the different processes. If a diagnostic test in subtraction of whole numbers shows that the student is unable to subtract in the example on the right, he should use his exploratory materials to discover the procedure to follow. He

$$\begin{array}{r} 2\ 0\ 3 \\ -1\ 5\ 6 \\ \hline \end{array}$$

would represent 203 as 2 hundreds and 3 ones. Then he would regroup the number as 1 hundred, 10 tens, and 3 ones. Finally, he would regroup this number as 1 hundred, 9 tens, and 13 ones. Then he could take away squares and strips representing 1 hundred, 5 tens, and 6 ones. He should continue to use objective materials and/or visual representations of the processes until he is able to perform the work with symbols alone. If he understands the algorisms for addition, subtraction, and multiplication, but not for division, he should use his exploratory materials for division but not for the other processes.

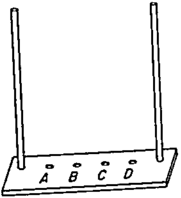
Each student should keep his exploratory materials in a suitable manila envelope, or preferably a small box. He may use the materials to ' ·lp him attain a minimum achievement in

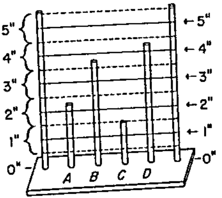
the basic processes. Part of this work must be done in the class period when other members of the class may be working on some unit or topic which is not part of the minimum program. Most students in the upper grades understand the meaning of the basic processes, hence they do not need much work with exploratory materials in dealing with whole numbers. However, in the upper grades there are many students who need exploratory materials in dealing with common and decimal fractions. These materials will be discussed in Chapter 6. The student should use objective materials only if he does not understand the work when it is presented with visual or symbolic materials.

Using Round Numbers

When we give the approximate distance from A to B as 300 miles or the approximate attendance at a football game as 40,000, we use *round numbers*. Round numbers are approximations of the true amount. These approximations should be sufficiently precise to enable a person to make a sensible interpretation of the given quantity. It is much easier to think intelligently with round numbers than with exact numbers. To illustrate, if the distance, to the nearest mile, between two places is 289 miles, this distance for convenience can be rounded off and expressed as 300 miles. It is easy to estimate that the driving time at 50 miles per hour would be approximately 6 hours, or that the gasoline consumption at 15 miles per gallon would be approximately 20 gallons. When 289 miles is used as the distance between two places, the difficulty of finding the estimated time of travel and the estimated gasoline consumption is greatly increased. Literacy in number is revealed by the ability to use round numbers intelligently.

Frequently a teacher gives instruction in the method of rounding off numbers and then has the student practice the process. Such a treatment of round numbers is of little value as a means of enriching a student's knowledge of number. He should use round numbers as a means of checking the reasonableness of an answer. A student should not be taught how to round off numbers merely to acquire skill in rounding off numbers.





The function of round numbers is to enable the student to grow in his ability to interpret quantities. Round numbers do not constitute a topic to be taught during a given week in the school program. Instead, they should be used throughout the course in arithmetic to enable the student to test whether an answer he finds to a problem or example is sensible. This should especially be true of the work in the upper grades. Buswell found that many students in high school and college were unable to make a reasonable estimate of the answer and that many used methods of estimating that were time-consuming.[5]

How to Round Off Numbers

Most students in the upper grades know the mechanics of rounding off numbers, but many of them do not understand what these numbers mean.

The picture shows a device which is an effective teaching aid to enable a student to discover the rules governing rounding off measures and numbers. This device consists of a wooden block which supports two vertical rods about a foot apart. These rods are divided into inches and half inches and are connected at each point of division by string so as to form parallel lines. The string connecting the points at the inch divisions is of a different color from the string at the half inch divisions. Holes in the block as shown support small rods of different lengths. The

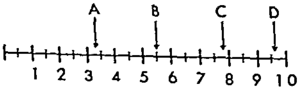
[5] Buswell, G. T. *Patterns of Thinking in Problem Solving*, p. 134. Berkeley, Calif.: University of California Press, 1956.

student should determine the length to the nearest inch of each of these rods. The length of rod A is more than 2 inches but it is less than $2\frac{1}{2}$ inches. Hence, its length to the nearest inch, is 2 inches. The length of rod B is between 3 inches and 4 inches, but it is nearer 4 inches than 3 inches, hence, the length would be expressed as 4 inches. Rod C terminates on a line connecting the midpoints between two inch marks. In this case the teacher must tell the student how to express the value. In rounding off numbers, it is conventional to give the value of the midpoint between two numbers the value of the greater number. Therefore, the length of rod C would be expressed as 2 inches.

Next, the student should express the length of each rod to the nearest half inch. He should use the frame to help him discover the rule for expressing a measurement to the nearest inch or half inch.

The scale can be changed from inches to represent feet, yards, miles, or any other unit of measurement. Similarly, it can represent a number. The values represented on the scale can be multiplied by 10 so that the given values would be 10, 20, 30, and similar values. Similarly, these scale values can be multiplied by any power of 10. The teacher should have the student use objective material of this kind until he discovers how to round off a measurement or a number.

Next, the teacher should have the student read the round numbers from a visual representation of a number scale of the kind shown below.



He should identify the values of points indicated. From illustrations of this kind the student should soon discover the procedure to follow for rounding off whole numbers. A whole number may be rounded off to the nearest 10, 100, 1000, or to any other power of our number base. The power of the base to which a number is to be rounded may be designated the unit

of measurement to which a given number is to be expressed. To round off a number, we proceed as follows:

1. *Identify the given number between the next lower and next higher round numbers in the given unit on the number scale.* Thus, to the nearest 100, 7438 would be between 7400 and 7500. To the nearest 1000, 7438 would be between 7000 and 8000.

2. *Assign to the given number the value of the rounded number to which the given number is nearer.* Thus, to the nearest 1000, 7438 would be assigned the value 7000 because 7438 is nearer to 7000 than to 8000, but 7538 would be assigned the value 8000.

3. *If a number is midway between two rounded numbers, assign to the given number the value of the greater rounded number.* Thus, 4500 is midway between 4000 and 5000. In this case, to the nearest 1000, the value of 4500 would be expressed as 5000.

The student who is learning how to round off whole numbers should follow the procedure given above. After he has developed more insight into the process, the following rules for rounding off whole numbers may be formulated:

1. *Use a zero to replace each figure dropped in rounding off a number.*
2. *When the first figure to be dropped is 5 or more, increase by 1 the next figure to the left.*
3. *When the first figure to be dropped is less than 5, drop that figure and all figures to the right of that figure.*

The number of figures to be retained in a number to be rounded off depends upon the use to be made of the round number. For most purposes, the two figures to the left in the number should give the degree of precision needed in the round number. For quick approximation, the first figure to the left in the number should be adequate for this purpose.

Diagnosis in Addition

We may assume that the student in the junior high school knows that addition is the process of finding the total number when two or more similar groups are joined. In order to perform the addition process with skill and competency, the student must:

1. Know the basic facts
2. Be able to add by endings

3. Know how to carry
4. Know that only figures in like places can be added
5. Be able to check the sum for accuracy and to use approximation to prove that his answer is sensible.

The reader may not be familiar with the meaning of adding by endings. In this form of addition, it is necessary to add a two-place number and a one-place number *in one* mental response. In the example on the right, the basic fact used is 7 + 6 = 13. Then the remainder of the addition in the column represents adding by endings. The combinations involved are 13 + 3 = 16 and 16 + 5 = 21. The thought pattern to follow in adding downward in the column should be, "13, 16, 21."

$$\begin{array}{r} 7 \\ 6 \\ 3 \\ \underline{5} \end{array}$$

When a student hesitates noticeably between consecutive figures in adding a column, makes a mark of some kind along the figures in a column, or uses his fingers or other objective means of counting to find a sum, he is having difficulty with adding by endings. Practice with examples of the types, 4 + 3, 14 + 3, 24 + 3, or 7 + 6, 17 + 6, and 27 + 6, should help the student to discover the relationship between a basic grouping and the ending of that grouping in the various decades. Examples should be written in sequential form as shown until the student is able to make this discovery. At practice time these examples should be arranged in random order.

In the example on the right, the sum should be found by adding downward in the columns and checked by adding upward. At the highest level of operation the student should use round numbers for finding the approximate sum. He should find the sum of 42, 1, 2, 1, and 7 to be 53, hence the approximate sum would be 5300. Therefore, the answer given is sensible. It is apparent that many other sums, such as 5271 or 5246 would be sensible but not accurate. One purpose of using round numbers in approximation of a sum is to detect glaring errors, such as giving 4281 or 6281 as the sum of the above example.

$$\begin{array}{r} 734 \\ 89 \\ 157 \\ 65 \\ \underline{4236} \\ 5281 \end{array}$$

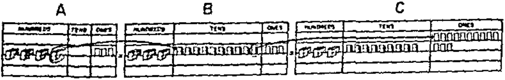
In order for approximation to be effective, it must be easy to estimate the answer. When addends have the same number of

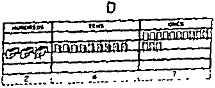
places, the student should find the sum of rounded figures given in the column on the left in the example. When addends have an unequal number of places, he should find the sum of the rounded figures in, at most, two columns on the left.

Diagnosis in Subtraction

Just as it usually can be assumed that the student in the advanced grades knows the meaning of addition, so it can be assumed that he knows the meaning of subtraction. He may encounter difficulty in compound subtraction, as in the example, $\begin{array}{r}421\\ -156\\ \hline\end{array}$ or $\begin{array}{r}4002\\ -1739\\ \hline\end{array}$. If a student does not understand the principle of regrouping the minuend, he should be able to discover the procedure to follow by use of the squares which are part of his arithmetic kit. A study of a visual representation of a few examples should enable the student to understand how to regroup the minuend in compound subtraction. The diagram gives a visual representation of the example on the right. The student should explain each step in the sequence. The teacher can show the process by use of a place-value chart.

$$\begin{array}{r}4\ 0\ 3\\ -1\ 5\ 6\\ \hline\end{array}$$





	6	7	8	9
6				
7				
8				
9				

Often a student who knows how to find the answer to a number grouping in multiplication, such as 6×8, is not certain of the facts involving the 6's, 7's, 8's, and 9's. The student should supply the missing products in a table of the kind shown. He should make study cards for facts not known. The study card shown for the grouping, 4×7, represents the type of study card which the student should make for the multiplication facts not known. It should be assumed that a student must be able to discover relationships among the basic facts. Thus, if he knows that $5 \times 8 = 40$, then he should be certain of the fact, $6 \times 8 = 48$ or $(5 + 1)8$. Since $10 \times 8 = 80$, 9×8 would be 8 less than 80, or 72.

Adding by Endings in Multiplication

A major source of error in multiplication for the slow learner is adding by endings. This process is necessary when the number carried must be added to a two-place product. The first column of Table VII gives all of the products of two one-place numbers beginning at 36. The next column gives the largest factor of each product, and the middle column gives the greatest number which can be carried to each product when multiplying by a one-place number. Thus, the product 36 may be found by multiplying 6 and 6, or 9 and 4. Therefore, the largest one-place factor of 36 is 9. When multiplying by 9, the largest carry number added to 36 is 8, hence it may be necessary to carry to 36 any number from 1 to 8. The student must be able to give the sums of the following examples which require adding by endings: $36 + 1$, $36 + 2$, $36 + 3$, $36 + 4$, $36 + 5$, $36 + 6$, $36 + 7$, and $36 + 8$.

In the table the two columns on the right group the examples into two classifications. In one group the sum does not span or bridge the decade, but in the other group the sum bridges the

Table VII. Multiplication Products Beginning with 36 and Number of Adding by Endings which May Be Formed

Product	Largest Factor	Greatest Carry Number	Adding by Endings with Bridging	Adding by Endings with No Bridging
36	9	8	5	3
40	8	7	0	7
42	7	6	0	6
45	9	8	4	4
48	8	7	6	1
49	7	6	6	0
54	9	8	3	5
56	8	7	4	3
63	9	8	2	6
64	8	7	2	5
72	9	8	1	7
81	9	8	0	8

decade. The endings 36 + 1, 36 + 2, and 36 + 3 do not bridge the decade. The five other endings, beginning with 36 + 4, bridge the decade. The group involving bridging usually is the source of more errors than the other group. The student who has difficulty with adding the carry number when multiplying by the numbers 6 to 9, inclusive, should be given special practice with examples which may be made from the data in the table. The teacher must be careful not to teach only *specifics*. A basic fact learned in isolation is an example of learning a specific.

Multiplying by a Two- or More-Place Number

Multiplying by 34 is the same as multiplying by the sum of 30 and 4. The part of the algorism involving multiplying by a two-place number which may not be understood is the placement of the second partial product. Exploratory materials to objectify the process should not be used unless the multiplier is a one-place number. The student

```
   5 6        5 6     5 6
 × 3 4      × 3 0     × 4
 -----    -------   -----
 2 2 4    1 6 8 0   2 2 4
1 6 8 0
-------
              2 2 4
          + 1 6 8 0
          ---------
```

should know that multiplying a whole number by 10 is the same as annexing a zero to the number. In the illustration the multipliers are 4 and 30. Since 30 is a multiple of 10, the product of 56 and 30 is 3 times the product of 560 (ten 56's). Therefore, the terminal zero is written. However, for most students in this age group, it should not be necessary to write the terminal zeros as place holders in the partial products. In the example on the right, the zeros in the two vacant places in the second partial product have been omitted. The student should understand that the partial product 846 represents 846 hundreds.

```
  4 2 3
 ×2 0 5
  2 1 1 5
8 4 6
```

The students who show average or above average ability in dealing with numbers should be taught to approximate the product. The method to use can be illustrated in the example on the right. Round off the multiplier as 200. Multiply 200 × 700, giving a product of 140,000. Since 736 is greater than 700 and 243 is greater than 200, the product of 736 and 243 must be greater than the product of 700 and 200, or 140,000. An approximation of the kind given will help the pupil to detect errors resulting from misplacement of a partial product, as illustrated in the example on the right. To approximate the answer, the student should think, "200 × 600 is 120,000." The approximate answer should prove to the student that the answer given to the example is not sensible. The function of checking an answer by use of round numbers by the method shown is twofold: First, to help able pupils to detect glaring errors of computation and second, to enrich the student's background of methods for dealing intelligently with numbers.

```
    736
   ×243
   2 208
  29 44
 147 2
 178,848
```

```
   586
  ×204
  2 344
 11 72
 14,064
```

Checking Multiplication

The most widely used check for multiplication is to go over the work. In the upper grades, the student who is making average or better achievement in arithmetic should be taught how to

check multiplication by *casting out nines*. Some teachers use this method to check the accuracy of computation in addition and subtraction. It seems absurd to check these processes by casting out 9's as the application of the check is more difficult than the process itself. On the other hand, casting out 9's is a very good check for multiplication.

Checking by casting out nines in a number is based on the *excess of nines* in that number. The excess of 9's is the remainder resulting from dividing a number by 9. Thus, in 38, the excess of nines ($9\overline{)38}$) is 2, but in 36 the excess of nines ($9\overline{)36}$) is 0. The excess of nines in a number may be found by adding the digits of the number and dividing the sum by 9 as well as by dividing the number by 9. In the number 36, the sum of the digits is 9. Of course 9 is a multiple of 9, hence there is no excess of nines in 36. If a number is divisible by 9, the sum of its digits is 9 or a multiple of 9. In the number 38, the sum of the digits ($3 + 8$) is 11 and in the number 11 the sum of the digits ($1 + 1$) is 2; also the remainder from dividing 38 by 9 is 2. If a number is not divisible by 9, the sum of the digits of that number is neither 9 nor a multiple of 9. If the sum is any one-place number, except 9, that sum is the excess of nines in the number. If the sum is a two-place number, such as 22, the excess of the nines in that number or the sum of its digits will be the excess of the nines in the given number. Thus, in the number 7384, the sum of the digits ($7 + 3 + 8 + 4$) is 22. The sum of the digits in 22 is 4 which is the excess of 9's in 22 and also in the number 7384. In the number 7326, the sum of the digits is 18, a multiple of 9; therefore, the excess of 9's in 7326 is zero.

Any two-place number is 10 times the value of the figure in tens' place plus the value of the figure in ones' place. Thus, 46 is equal to $10 \times 4 + 6$. If we represent the tens' digit by t and the units' digit by u, any two-place number may be represented as $10t + u$. The value of $10t + u$ may be expressed as $9t + t + u$. The term $9t$ is a multiple of 9, therefore, the remainder from dividing $9t + t + u$ by 9 is $t + u$, which represents the sum of the digits.

The product of the excesses of nines in two numbers multiplied is equal to the excess of nines in the product of the numbers.

We shall assume the truth of this theorem as its proof is beyond the scope of junior high school mathematics. In the example on the right, the numbers circled represent the excess of 9's in the two numbers multiplied. The product of the excesses is 35 and the excess of 9's in 35 is 8 (see number in square). The excess of 9's in the answer (31562) of the example is 8, therefore we may be reasonably certain that the example is correct. When the answer does not check by casting out 9's, it is certain that the work is not correct. The reader who is not familiar with this method of checking by casting out 9's should multiply and check the product by this method in the following examples:

	Excess of 9's
734	5
×43	×7
31,562	35
↓	↓
$\boxed{8}$ =	$\boxed{8}$

$$\begin{array}{r} 87 \\ \times 46 \\ \hline \end{array} \qquad \begin{array}{r} 902 \\ \times 57 \\ \hline \end{array} \qquad \begin{array}{r} 1726 \\ \times 47 \\ \hline \end{array} \qquad \begin{array}{r} \$18.75 \\ \times 48 \\ \hline \end{array}$$

Casting Out 9's Not Always a Valid Check

Casting out nines will not reveal an error resulting from misplacing a figure in a product. The number 31,562 could be written as 31,652 or any other arrangements of these digits and the answer would check. In the example 204 × 586 on page 157 the answer given was 14,064. In this case the figures of the second partial product were written in wrong places. The number 586 was multiplied by 24 instead of 204. The difference between 204 and 24 is 180, a multiple of 9, therefore, the incorrect product, 14,064, checks by casting out nines.

After the student becomes familiar with the method of finding the excess of 9's in a number, he should discover shortcuts to use. To find the excess of 9's in 72,934, he would add 7 and 2 and drop the sum since the excess of nines in 9 is zero. He would skip the 9 and add 3 and 4. The excess of 9's in 72,934 is 7. In the example on the right, the excess of 9's in 342 is zero, therefore, it is not necessary to find the excess of 9's in 138 because the product of zero and any number is always zero. The excess of

$$\begin{array}{r} 342 \\ \times 138 \\ \hline 46,296 \end{array}$$

9's in the product must also be zero if the answer is correct. The sum of the digits 4, 6, 2, and 6 is 18, and the sum of the digits in 18 is 9, hence the excess of nines in the product is zero. Very probably the answer is correct. The student who learns to check by casting out 9's not only has a practical social application of number, but also learns an interesting property of the number 9, which is one less than the base of our number system. The latter feature is not to be minimized in a program which stresses growth and understanding in arithmetic.

Division with a One-Place Divisor

There are very few educational problems which can be answered with complete assurance that the answer given is correct. It cannot be maintained that the long form of division with a one-place divisor is the only method to teach, especially in a program which stresses meaning and understanding. On the other hand, all published experiments in this field have shown that the long form is superior to the short form. In the long form all work is displayed, but in the short form only the quotient is written. In three different research investigations, Grossnickle[6] showed that the long form is superior to the short form. No published investigation has ever shown that the short form is as good as or superior to the long form, even for students at the high school or college level.

Many teachers in junior and senior high school believe that the student at these levels should use the short form of division when the divisor is a one-place number. It should be understood that there must be carrying from one partial dividend to the next partial dividend in order to have so-called "long division." Long division is neither possible nor desirable in the example,

[6] Grossnickle, Foster E. "An Experiment with a One-Figure Divisor in Short and Long Division," *Elementary School Journal*. 34:496–506; 590–599. Chicago: University of Chicago Press.

——— "Appraising the Program for Teaching Division," *National Elementary Principal*. 16:361–368.

——— "The Incidence of Error in Division with a One-Figure Divisor," *Journal of Educational Research*. 29:509–511.

$3\overline{)906}$, because each partial dividend is seen. On the other hand, it is possible and usually desirable to use this form of division in the example, $3\overline{)732}$, because there is uneven division (remainders) when dividing the 7 and the 3, and the products are unseen. The writers recommend that all students learn to use the long form to divide by a one-place number in uneven division. A student later may be encouraged to use shortcut procedures and thereby learn the short form. Many students at the junior or senior high school level can learn to use the short form successfully.

The problem of the form of the algorism centers on the relative merits of meaningful and mechanical processes. The method used does little to alter the student's understanding of the process. On the other hand, the ability to approximate a quotient to see if it is sensible shows growth in ability to deal with division. It is more important from the standpoint of growth in ability to deal with number for the student to be certain that an answer is sensible than to worry about the method used in performing the algorism to get the answer.

A recent study[7] showed that the conventional way of dividing is not the best method from the standpoint of teaching meaning and understanding of the process. The experimental evidence proved that the method based on subtraction was more effective for slow learners than the conventional method. Thus, in the example shown, the student would find the quotient to be the sum of 20 + 5 + 4, or 29, with a remainder of 5. Most students at the junior high school level should be able to operate at a more mature level than shown.

$$
\begin{array}{r|r}
8\overline{)237} & 20 \\
\underline{160} & \\
77 & 5 \\
\underline{40} & \\
37 & 4 \\
\underline{32} & \\
5 &
\end{array}
$$

Checking the Quotient by Approximation

The example on the next page illustrates a common type of error in division. The student may use the conventional

[7] Van Engen, Henry and Gibb, E. G. *General Mental Functions Associated with Division*. Cedar Falls, Iowa: State Teachers College, 1956

form of checking by multiplying quotient by divisor and adding remainder to that product and not detect the error shown in the illustration. On the other hand, the student should use approximate numbers to estimate the answer. He should think of 4553 as 4600. Then his thought pattern would be as follows: "The quotient must be more than 600 but less than 700." Then the given answer, 65 r3, is not sensible. The method used in the algorism is predominantly a mechanical process. The estimation to show that the answer is sensible represents a level of quantitative thinking which most students in the upper grades should be able to attain.

```
   65 r 3
7)4553
  42
   35          Check:
   35            65
    3            ×7
                455
                + 3
               4553
```

In light of experimental evidence, it seems prudent to have the student use the form which has been demonstrated consistently to be the superior form for performing the algorism. For purposes of differentiating the curriculum, superior students should be encouraged to use the short form. Regardless of the method used, most students in the upper grades should be instructed to use round numbers to approximate the quotient. The slow learners should not be expected to master this procedure, except in solving easy examples. All students but the slow learners should be able to see that the quotient of the example, $8\overline{)5136}$, must be between 600 and 700. A program which fails to develop the student's ability to deal with this phase of division does not emphasize growth in literacy in the use of number.

Most students who have completed the intermediate grades know how to divide by a one-place divisor and how to check the work by multiplying quotient by divisor and adding the remainder to that product. In the upper grades, the student should not merely review the method by solving more division examples, but also he should develop skill in using round numbers to check quotient figures by making an approximation of the true quotient. He should show growth in ability to use the division process by gaining confidence that the answer is both correct and sensible.

Using a Diagnostic Test in Division by a Two-Place Divisor

A diagnostic test of the kind shown on page 133 will show where students have difficulty with division by a two-place divisor. The test will show where this skill breaks down, thus indicating lack of understanding and inability to proceed. These students should be taught division at the level at which they will understand the work. The other students who encountered no difficulty on the diagnostic test should be taught how to use round numbers to check the reasonableness of an answer.

Methods of Estimating the Quotient

There are approximately a dozen different methods of estimating the quotient, but the two most widely used may be designated as the *apparent method* and the *increase-by-one method.* According to the apparent method, the *guide figure* (tens' figure of a two-place divisor) remains unchanged regardless of the figure in ones' place. Thus, the guide figure of the divisors 34 and 37 is 3. According to the increase-by-one method, the guide figure remains unchanged when the figure in ones' place is 5 or less, but the guide figure is increased by one when the figure in ones' place is 6 or more. Thus, the guide figure of the divisor 35 is 3, but the guide figure of the divisor 36 is 4.

The student who has difficulty in dividing by a two-place number should begin to restudy the process by using a divisor of 10. If necessary, he may use objective materials to find the number of 10's in a given number, such as 40 or 43 and then the number of 20's in the same number. He should soon discover that the number of 20's in a number, such as 67, is the same as the number of 2's in 6.

From the standpoint of estimation of quotient, the most difficult part consists in finding the true quotient when the estimated quotient has to be corrected,[8] as in the example, $37\overline{)162}$. Excluding the divisors in the teens, the estimated quotient by the apparent method may be three more than the true

[8] See Brueckner, L. J. and Melbye, H. O. "Relative Difficulty of Types of Examples in Division with Two-figure Divisors," *Journal of Educational Research.* 33:401–413.

quotient, as in $28\overline{)183}$. There the estimated quotient is 9 ($2\overline{)18}$), but the true quotient is 6, or three less than the estimated figure. If the estimated quotient must be corrected to find the true quotient, the estimated figure is made smaller. When the student is taught to use the increase-by-one method, he will find that the estimated quotient sometimes must be decreased, as in the example $34\overline{)180}$, and sometimes the estimated quotient must be increased, as in the example, $26\overline{)234}$.

The student should use judgment in estimation, especially when the estimated quotient is 8 or 9, by first multiplying the divisor by 10. In the example, $28\overline{)183}$, the estimated quotient by the apparent method is 9, but $10 \times 28 = 280$, hence 183 cannot be 9 times as large as 28. The student should try 6 or 7.

For divisors 13 to 19 in the teens and for divisors 26 to 29 in the twenties, the slow-learning student should make a table for a given divisor and use it to discover the quotients. The estimated quotients for these divisors frequently need to be corrected one or more times to find the true quotient. The percentage of corrections for the divisors in the other decades is considerably smaller than for the divisors mentioned in the 10's and 20's. This is true because of the relative values of the figures in ones' and tens' places in a two-place divisor. As the value of the figure in tens' place increases in a two-place number, the relative value of the figure in ones' place decreases. The relative value of the 6 ones to 1 ten in the number 16 is much greater than the relative value of the 6 ones to 7 tens in the number 76. The per cent of estimations which will be incorrect for the divisor 16 is much greater than for the divisor 76. Similarly, the per cent of estimations which need to be corrected for the divisors in the tens and twenties will be greater than the corresponding per cent of estimations for the divisors in the higher decades.

Part of the table for the divisor 17 is shown. Thus at a glance the student can see that the quotient of $17\overline{)49}$ is between 2 and 3. If a student has difficulty in finding the quotient for divisors in other decades, this method should be used. The method is long and not efficient,

$$1 \times 17 = 17$$
$$2 \times 17 = 34$$
$$3 \times 17 = 51$$
$$\cdot$$
$$10 \times 17 = 170$$

but the slow learner will understand it. This latter feature is more important than learning by rote a more efficient process which will soon be forgotten after a short period of disuse.

A table of the kind shown on the right requires less time to construct than a complete table for a divisor, as 17. The values 1, 2, 4, and 8 are successive doubles. Thus, the tabular value for 8 will be twice the tabular value for 4, or 2 × 68. The student should know 10 times the divisor, or 10 × 17 = 170. Then the value of 5 times the divisor will be half the tabular value of 10 times that number.

1	17
2	34
4	68
5	85
8	136
10	170

Regardless of the method of estimation used in finding the quotient for the divisors 13 to 19, the estimated quotient often has to be corrected many times to find the true quotient. The student should approximate the quotient by first finding 10 times the divisor and then 5 times the divisor. Of course, 5 times the divisor will be half the product of 10 times the divisor. If the divisor is 16, the student should think, "10 × 16 = 160, then 5 × 16 will be $\frac{1}{2}$ of 160, or 80." To estimate in the example, $16\overline{)102}$, the student should see that 5 × 16 = 80, and that one more 16 would be 96, hence, the quotient would be seen to be 5 + 1, or 6.

At the junior high school level, a student who demonstrates skill in dividing by a two-place number should use round numbers to check the quotient. He may approximate the answer before he finds the true quotient, or he may use round numbers to check the answer he has found. Since the student knows the algorism, it is no longer important whether he uses the apparent method, the increase-by-one method, or some modification of either method or of both. He should be able to determine whether or not the answer is sensible. In the example, $46\overline{)2753}$, the student should think of 2753 as 2800. The quotient must be between 10 and 100 because 10 × 46 = 460 and 100 × 46 = 4600. Now he should think that the divisor is between 40 and 50. If the divisor were 40, the quotient would be 70; if the divisor were 50, the quotient would be a little more than 50. The quotient must be between 50 and 70. Any answer within

that range would be sensible. It is entirely satisfactory for purposes of approximation to have the student make only one estimation of the approximate value of the quotient. In that case he could either estimate the number of 40's in 2800 or the number of 50's in 2800. Therefore, the student should be able to approximate the true quotient to be less than 70 or more than 50.

Checking Answers in Division

One widely used method of checking the accuracy of the work in division is to multiply divisor and quotient and add the remainder to that product. Another check is to go over the work. Casting out 9's may be used, especially for purposes of enrichment of the curriculum for the superior students. If this check is used, the final remainder first should be subtracted from the dividend. The resulting number is a multiple of the divisor. In the example on the right, the quotient is 5 r7. If 7 is subtracted from 187, the answer, 180, is a multiple of 36. Then the check for casting out 9's is the same in division as for an example in multiplication.

$$\begin{array}{r} 5 \text{ r}7 \\ 36\overline{)187} \\ \underline{180} \\ 7 \end{array}$$

Check:

	Excess of 9's
36	0
×5	×5
180	0

Grossnickle[9] made an intensive study of errors made by a group of 221 pupils as they learned how to divide by a two-place divisor. These pupils made a total of 113 different kinds of errors, but only those errors were persistent which resulted from estimation and from faulty answers to combinations. The study showed that nearly all errors were due to chance and they could be corrected by a careful analysis of a pupil's work. A close scrutiny of a student's own work, supplemented by his use of checking the work should enable the student to attain a minimum satisfactory achievement in division with a two-place divisor.

[9] Grossnickle, Foster E. "Constancy of Error in Learning Division with a Two-Figure Divisor," *Journal of Educational Research.* 33:189–196.

Division with a Three-Place Divisor

Division is a complex process since it involves the use of the other three processes. Because of its complexity and difficulty, it should be spaced through several grades or spiraled for different age levels. Division with a three-place divisor should be introduced at the upper level of the elementary school. Many examples in division with a three-place divisor, as in the example, $312\overline{)936}$, are less difficult than some examples with a two-place divisor, as in the example, $16\overline{)112}$, but the social usage of a three-place divisor is limited for most students in the elementary school. The infrequent social applications of division with a three-place divisor justifies deferring the teaching of the process.

The student should understand three things about division: (1) the meaning of the process; (2) the correct placement of the first quotient figure; and (3) how to estimate the quotient. The placement of the first quotient figure is the one element which is new to the student when he first divides by a three-place divisor. He learned the meaning of division when dealing with a one-place divisor and how to use the guide figure in estimation of the quotient with a two-place divisor. When the divisor is a three-place number, the guide figure is the figure in hundreds' place. This guide figure is treated in the same manner as the guide figure of a two-place divisor.

Placement of the Quotient Figure

Either of two ways, or both, may be used in order to have the student understand where to place the first quotient figure when the divisor is a three-place number. According to one plan, the student should determine the place of the first quotient figure by use of the ungrouped value of the dividend. In the example on the right, the thought pattern for finding the quotient figure should be as follows: "In 982 there are 98 tens. Since 98 is less than 314, change 98 tens to ones, making a total of 982 ones. The number of 314's in 982 will be the same as the number of 3's in 9, or 3. Write 3 in ones' place in the quotient."

$$\begin{array}{r} 3 \\ 314\overline{)982} \end{array}$$

According to the second plan, the first figure of the quotient is found by multiplying the divisor by 10 or a power of 10. In the example given above, 982 is greater than 314, therefore, the quotient must be at least 1. But $10 \times 314 = 3140$, therefore, the quotient must be less than 10, or a one-place number. The number of 314's in 982 is the same as the number of 3's in 9, or 3, which is written in ones' place in the quotient. In the example, $215\overline{)11,468}$, the student should think, "$10 \times 215 = 2150$ and $100 \times 215 = 21,500$, then the quotient must be more than 10 but less than 100, hence the quotient must be a two-place number. To find the figure in tens' place in the quotient, think $2\overline{)11}$, or 5."

The student should form the habit of checking the number of places in the quotient by multiplying the divisor by 10 or a power of 10 to see that the answer is sensible. Then the type of frequent error illustrated by the example on the right can be eliminated. In this example, the student should see that the quotient must be more than 100 because $100 \times 243 = 24,300$, which is less than 82,625.

$$\begin{array}{r} 34 \text{ r } 5 \\ 243\overline{)82,625} \\ \underline{729} \\ 972 \\ \underline{972} \\ 5 \end{array}$$

Method of Estimation for a Three-Place Divisor

The student should use the apparent method of estimation of the quotient when the divisor is a three-place number until he learns the process. This is the procedure followed when the divisor is a two-place number. From that point, he may discover a method which is a modification of the apparent and increase-by-one methods, or use some other individual method. After he understands the process and the means for verifying an answer by approximation, the method to use to perform the algorism is of minor significance. Of course, he learns to correct the estimated quotient as he did when dividing by a two-place divisor. In the example, $365\overline{)1672}$, the student sees that the quotient is less than 10 because 10×365 is greater than 1672. He estimates $3\overline{)16}$ which is 5, but 5 is too large, hence the

estimated figure is made one less, or 4. One of the great advantages of the *apparent method* of estimation is the uniform procedure of decreasing the estimated figure to find the true quotient figure when the estimated figure is incorrect.

Many teachers do not know how to introduce division by a three-place divisor to the slow-learning student. Two different solutions may be given to this problem. One plan is to delete this material from the curriculum for the slow learner because he will have, at most, a very limited need for this type of division. The other plan is to have the student use a method of division which he understands although it may not be as efficient as the conventional method. If he is not able to estimate the quotient figure for a divisor, he should make a table similar to the table for 230 as shown on the right. In initial learning of the subject, the examples must be carefully graded so as to present a minimum of difficulty in learning, and should be limited to one-place quotients.

$$
\begin{aligned}
2 \times 230 &= 460 \\
3 \times 230 &= 690 \\
4 \times 230 &= 920 \\
5 \times 230 &= 1150 \\
&\;\;\vdots \\
9 \times 230 &= 2070
\end{aligned}
$$

e. *Discovery of Relationships among Processes*

The teacher who stresses meaning and understanding of number will provide many opportunities for students to discover relationships between addition and subtraction and between multiplication and division. The student should be able to identify the two examples in addition and the two in subtraction which can be made from the numbers in box A. In the same way he should discover that two examples in multiplication and two in division can be made from the numbers in box B.

A	16	24	40
B	12	15	180

By inspection the student should be able to rank the examples given below in order of the size of the sums without adding.

a.	b.	c.	d.	e.
125	125	125	125	125
+236	+426	+103	+370	+245

Then he should add the numbers to see if his ranking of the examples was correct. A modification of the above exercise consists in ranking the examples according to the size of the missing numbers before finding the missing numbers:

1. a. $\begin{array}{r} 500 \\ -138 \\ \hline ? \end{array}$ b. $\begin{array}{r} 500 \\ -346 \\ \hline ? \end{array}$ c. $\begin{array}{r} 500 \\ -250 \\ \hline ? \end{array}$ d. $\begin{array}{r} 500 \\ -107 \\ \hline ? \end{array}$ e. $\begin{array}{r} 500 \\ -435 \\ \hline ? \end{array}$

2. a. $\begin{array}{r} 500 \\ ? \\ \hline 182 \end{array}$ b. $\begin{array}{r} 500 \\ ? \\ \hline 250 \end{array}$ c. $\begin{array}{r} 500 \\ ? \\ \hline 119 \end{array}$ d. $\begin{array}{r} 500 \\ ? \\ \hline 365 \end{array}$ e. $\begin{array}{r} 500 \\ ? \\ \hline 416 \end{array}$

Then the student should solve the examples to check his ranking. One of the principles under subtraction on page 143 states that if either of two numbers is subtracted from their sum, the remainder would be the other number. The superior students should be able to form this generalization from a study of the examples given above.

In the examples below, the student should be able to rank the examples according to the size of the products before multiplying.

a. $\begin{array}{r} 45 \\ \times 12 \\ \hline \end{array}$ b. $\begin{array}{r} 45 \\ \times 45 \\ \hline \end{array}$ c. $\begin{array}{r} 45 \\ \times 20 \\ \hline \end{array}$ d. $\begin{array}{r} 45 \\ \times 31 \\ \hline \end{array}$ e. $\begin{array}{r} 45 \\ \times 27 \\ \hline \end{array}$

The superior students should be able to form the generalization which applies to this series. This generalization may be stated as follows: If different examples have equal multiplicands but unequal multipliers, the larger the multiplier, the larger will be the product.

The following series of examples represent the type that may be used to help the student discover the relationship among divisor, dividend, and quotient. He should rank these examples in the order of the size of the missing numbers before solving for these numbers.

1. a. $60\overline{)360}$ b. $90\overline{)360}$ c. $30\overline{)360}$ d. $180\overline{)360}$ e. $15\overline{)360}$

2. a. $\begin{array}{r} 40 \\ ?\overline{)360} \end{array}$ b. $\begin{array}{r} 90 \\ ?\overline{)360} \end{array}$ c. $\begin{array}{r} 10 \\ ?\overline{)360} \end{array}$ d. $\begin{array}{r} 120 \\ ?\overline{)360} \end{array}$ e. $\begin{array}{r} 24 \\ ?\overline{)360} \end{array}$

These examples illustrate the mathematical principle, given on page 144, which states that if the product of two numbers and one of the numbers are given, the other number may be found by dividing the product by the given number. The very superior students should be able to make the generalization in regard to the value of the missing factor This generalization may be stated as follows: If different examples have equal dividends, the larger the divisor, the smaller will be the quotient as shown in row 1, or the larger the quotient, the smaller will be the divisor as shown in row 2. The teacher who has the student discover relationships of the kind shown in the multiplication examples or in the division examples lays the basic work for the study of variation in the first year of algebra. If two variables (numbers which change value) are in *direct variation*, their ratio remains constant. As one number increases, there is a corresponding increase in the other number.[10] In the multiplication examples given above, the multiplicand is the constant ratio The two variables in the examples are the multipliers and the products. As the multiplier increases, there is a corresponding increase in the product.

Two variables vary *inversely* when their product is constant. When one of the factors of the product increases in value, the other factor decreases in value.[10] The two sets of examples in division represent this principle. Therefore, the student who is helped in making discoveries about relationships among numbers is being prepared for further study in the field of mathematics. An acceptable program for meeting the needs of individual differences will explore the potential ability of students to do further study in mathematics.

[10] This applies only to positive numbers.

Questions, Problems, and Topics for Discussion

1. What constitutes mastery of a process, such as addition of whole numbers?
2. Outline a plan for testing a student's knowledge of the basic facts in subtraction.
3. Outline a plan to use for teaching the basic facts not known by a student.
4. What is meant by place value in our number system? How is it possible to have place value without the use of zero?

5. Express the following numbers in scientific notation: a. 7,250,000 b. .0000045 c. 9 billion.

6. Illustrate grouped numbers, ungrouped numbers, and regrouped numbers. Show that an ungrouped number becomes a grouped number as soon as the ungrouped number is written.

7. State in which of the following problems the division represents measurement and in which, partition:

a. How many gallons in 12 quarts?
b. How many 5-cent stamps can be purchased with a dollar?
c. How many yards in 15 feet?
d. A car used 4 gallons of gasoline in a trip of 60 miles. What was the average number of miles per gallon?

8. Give two principles which govern addition and also two principles which characterize subtraction. Give illustrations of each principle.

9. Give three basic principles which characterize multiplication and division. Illustrate each principle with whole numbers.

10. What is meant by the reciprocal of a number? Give illustrations of the use of reciprocal numbers.

11. What are round numbers? Write the rules which govern rounding off numbers.

a. Round off to the nearest thousand: 72,560, 34,490, 1500.
b. Round off to the nearest hundredth: .465, .7449, .995.

12. What is meant by adding by endings? Where is this form of addition used? Show how you would teach adding by endings.

13. Make a visual representation to show the steps in subtracting 348 from 516.

14. What is meant by teaching specifics? Illustrate the principle in addition or subtraction.

15. What is meant by the "excess of nines" in a number? Find the excess of nines in 7,348,912.

16. Many students who are lefthanded have a tendency to reverse the digits in a number, as in writing 42 for 24. Why will checking by casting out 9's not reveal an error resulting from the reversal of the digits in a number?

17. Write a four-place number, such as 8241, and subtract from it any other smaller four-place number made with the digits 8, 2, 4, and 1. Prove that the remainder is a multiple of 9.

18. Show why you would or would not teach division with a one-place divisor in the short form.

19. Illustrate the difference between the apparent method and the increase-by-one method of estimation of the quotient. List some of the advantages and disadvantages of each method.

20. Divide 4258 by 58 and check the work by casting out 9's.

21. Illustrate two different procedures for finding the position of the first figure in the quotient.

22. Show how you would introduce dividing by a three-place divisor to slow learners.

23. The areas of different rectangles are 60 square inches. Use different values to show how a change in one dimension affects the other dimension.

Suggested Readings

Beatley, Ralph "Reason and Rule in Arithmetic and Algebra," *The Mathematics Teacher*. 47:234–244.

Brueckner, Leo J. and Grossnickle, Foster E *Making Arithmetic Meaningful*, Chapters 6, 7, and 8. Philadelphia: The John C Winston Co, 1953

Clark, John R. and Eads, Laura *Guiding Arithmetic Learning*, pp. 59–134. Yonkers, N. Y.: World Book Co., 1954.

Grossnickle, Foster E. "Teaching Arithmetic in the Junior High School," *The Mathematics Teacher*. 47:520–527.

Morton, Robert L. *Teaching Children Arithmetic*, Chapters 4 and 5. New York: Silver Burdett Company, 1953.

Smith, Rolland R. "Meaningful Division," *The Mathematics Teacher*. 43:12–18.

Spitzer, Herbert F. *The Teaching of Arithmetic*, Chapters 4 and 5. Boston: Houghton Mifflin Company, 1954.

Van Engen, H. "Arithmetic in the Junior-Senior High School," *The Teaching of Arithmetic*, Chapter 6. The Fiftieth Yearbook of the National Society for the Study of Education, Part II Chicago. University of Chicago Press, 1951.

Van Engen, H. and Gibb, E. G. *General Mental Functions Associated with Division*. Cedar Falls, Iowa: State Teachers College, 1956.

Chapter 6

Growth in Dealing with Common and Decimal Fractions

The major topics treated in this chapter are:

a. The need of differentiating instruction in fractions
b. Teaching the basic operations with common fractions
c. Teaching the basic operations with decimal fractions
d. The kinds of problems in common and decimal fractions.

a. The Need of Differentiating Instruction in Fractions

Knowledge of Fractions Possessed by Students in the Upper Grades

Many students in the junior high school and at higher grade levels have a very fragmentary knowledge of common and decimal fractions as shown by achievement tests in the subject. From a study of the records made by 937 students in the ninth grade and 925 college freshmen, Guiler found the per cent of students having errors on a test in the four processes with common fractions was as follows:

Process	*Per Cent of Group Making Errors* *Ninth Grade*[1]	*College Freshmen*[2]
Addition of fractions	23.0	44.5
Subtraction of fractions	42.5	63.5
Multiplication of fractions	42.5	53.3
Division of fractions	40.7	57.7

[1] Guiler, Walter S. "Difficulties in Fractions Encountered by Ninth Grade Pupils," *Elementary School Journal*. 46 147 Chicago University of Chicago Press.
[2] ——— "Difficulties Encountered by College Freshmen in Fractions," *Journal of Educational Research*. 39: 103.

"Lack of comprehension of the process constituted the outstanding source of the difficulties encountered by the college freshmen in their work with fractions. This type of difficulty was particularly pronounced in the multiplication and division of fractions. In this category most difficulty was encountered in division and least difficulty in addition."[3] Apparently many students had learned fractions by dealing almost exclusively with symbolic materials which may have had little meaning to them. Under such conditions it was necessary for the student to learn by rote the principles governing the operations with fractions, hence he did not understand this work Due to lack of practice or disuse of the operations with fractions by college students, it was also true that the skill had deteriorated considerably. It is known that these skills can be rebuilt by a short period of well organized practice as determined by the application of procedures based on reliable diagnosis.

Finding Those Students Needing Help in Common Fractions

All students who enter the junior high school have had instruction in the basic processes in common and decimal fractions. As in learning of most topics in arithmetic, some of the students have much greater understanding and greater skill in computing with fractions than other students. The task of the teacher at the start of the year is to find the students who lack these skills and to take steps to correct the deficiencies. The results from a diagnostic test in fractions in each basic process should enable the teacher to find the place at which the student's understanding and computational skill are deficient. The proper remedial instruction can then be applied. The teacher also should find the students who demonstrate a high level of achievement in fractions and enrich their understanding of the topic.

The diagnostic test in subtraction of fractions given on page 176 is representative of the type of test to be used for each of the processes in this field. This test is divided into two parts. In Part I, the numbers of the minuend need not be regrouped, but

[3] *Ibid.*, p 114.

Diagnostic Test in Subtraction of Common Fractions

Part I. Numbers above Need Not Be Changed

	a	b	c	d	e	f
1.	$\frac{2}{3}$ $\underline{\frac{1}{3}}$	$\frac{5}{8}$ $\underline{\frac{1}{8}}$	$6\frac{3}{4}$ $\underline{5}$	$6\frac{7}{8}$ $\underline{1\frac{5}{8}}$	$4\frac{7}{12}$ $\underline{\frac{7}{12}}$	$7\frac{5}{8}$ $\underline{7\frac{3}{8}}$
2.	$\frac{3}{4}$ $\underline{\frac{1}{2}}$	$\frac{5}{6}$ $\underline{\frac{1}{2}}$	$\frac{11}{12}$ $\underline{\frac{3}{4}}$	$8\frac{3}{4}$ $\underline{8\frac{1}{2}}$	$9\frac{5}{6}$ $\underline{4\frac{1}{3}}$	$8\frac{2}{3}$ $\underline{\frac{1}{6}}$
3.	$\frac{2}{3}$ $\underline{\frac{1}{2}}$	$\frac{7}{8}$ $\underline{\frac{2}{3}}$	$\frac{2}{3}$ $\underline{\frac{1}{4}}$	$9\frac{3}{4}$ $\underline{4\frac{1}{3}}$	$5\frac{2}{3}$ $\underline{1\frac{3}{8}}$	$6\frac{2}{3}$ $\underline{\frac{1}{5}}$
4.	$\frac{7}{8}$ $\underline{\frac{1}{6}}$	$\frac{5}{6}$ $\underline{\frac{3}{4}}$	$\frac{7}{8}$ $\underline{\frac{5}{12}}$	$6\frac{3}{4}$ $\underline{4\frac{3}{10}}$	$8\frac{3}{4}$ $\underline{8\frac{1}{6}}$	$4\frac{7}{12}$ $\underline{\frac{3}{8}}$

Part II. Numbers above Must Be Regrouped

	a	b	c	d	e	f
5.	1 $\underline{\frac{2}{3}}$	2 $\underline{1\frac{3}{8}}$	8 $\underline{1\frac{3}{4}}$	5 $\underline{4\frac{5}{6}}$	8 $\underline{6\frac{3}{5}}$	11 $\underline{8\frac{3}{4}}$
6.	$7\frac{1}{3}$ $\underline{4\frac{2}{3}}$	$5\frac{1}{4}$ $\underline{2\frac{3}{4}}$	$8\frac{1}{8}$ $\underline{5\frac{7}{8}}$	$7\frac{1}{6}$ $\underline{4\frac{5}{6}}$	$12\frac{1}{4}$ $\underline{9\frac{3}{4}}$	$5\frac{1}{12}$ $\underline{\frac{7}{12}}$
7.	$9\frac{1}{2}$ $\underline{1\frac{3}{4}}$	$6\frac{1}{3}$ $\underline{5\frac{5}{6}}$	$9\frac{1}{2}$ $\underline{5\frac{7}{8}}$	$6\frac{1}{3}$ $\underline{4\frac{7}{12}}$	$8\frac{1}{5}$ $\underline{7\frac{9}{10}}$	$7\frac{3}{4}$ $\underline{\frac{11}{12}}$
8.	$7\frac{1}{2}$ $\underline{4\frac{2}{3}}$	$5\frac{1}{4}$ $\underline{4\frac{2}{3}}$	$9\frac{1}{2}$ $\underline{7\frac{4}{5}}$	$10\frac{3}{4}$ $\underline{6\frac{5}{6}}$	$8\frac{1}{6}$ $\underline{4\frac{7}{8}}$	$1\frac{1}{8}$ $\underline{\frac{5}{6}}$

in Part II, the numbers in the minuend must be regrouped in order to subtract.

The examples in each row of the test are similar and involve the same structural difficulty. To illustrate, consider the examples

in Part I, row 3. In each of these examples the lowest common denominator is the product of the denominators. In the next row the denominators in each example are unlike, but the lowest common denominator is not the product of the denominators.

If a student's test has only one or at most two examples incorrect in any row, very probably these mistakes were chance errors or due to a lapse of attention. On the other hand, if three or more examples in any row are incorrect, it is highly probable that the errors were due to a faulty procedure of some kind.[4] Then the teacher should determine through an analysis of the student's work or through a personal interview the nature of the errors. Proper remedial measures should then be applied to correct the errors. In many cases these errors may have resulted from a faulty knowledge of the operations themselves as shown by Guiler. In that case, the teacher should provide the kinds of experiences that will enable the student to understand fractions and how to compute with them. Since learning is an individual process, each student should be provided with the kinds of materials that will enable him to correct his deficiencies.

Kit for Fractions

A student whose understanding of fractions is very limited should learn to use fractional cut-outs to discover meanings. These cut-outs should be part of his arithmetic kit. These cut-outs can be made of either oaktag or cardboard. The cut-outs for fractions should consist of two whole circles and five pairs of circles cut into halves, thirds, fourths, sixths, and eighths. Each circle should be approximately 5 inches in diameter. The student in the junior high school should use compasses to draw the pattern and then cut the circles to form the different fractional parts. He should use a protractor to mark the circles to show thirds and sixths, or lay off arcs on the circle with a chord equal to the radius of the circle. A manila envelope or a small box is suitable for storing the cut-outs.

[4] Brueckner, L. J. and Hawkinson, Ella "Optimum Order of Arrangements of Items in a Diagnostic Test in Arithmetic," *Elementary School Journal*. 34:351–356. Chicago: University of Chicago Press.

The student who has squares and rectangular strips for his kit, as suggested on page 146, may use these materials as part of his kit for decimal fractions. The directions for the preparation of these squares stated that one side of the oaktag should be ruled and the other side should be blank. The blank side of the squares ruled to represent hundreds may be used to represent ones; the blank side of the strip ruled to represent tens may be used to represent tenths. Either side of the individual squares may be used to represent ones or hundredths. Any student who either does not understand decimal fractions or how to perform the algorisms with them should make a set of squares as described on pages 146–147. He should use these materials to discover relationships among decimal fractions and whole numbers.

The students use the parts of their fraction kit to review the meaning of fractions. The flannel board demonstration verifies the students' activity.

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Meanings of Fractions

A fraction conveys four different meanings: (1) a fraction may represent a part of a whole or (2) identify part of a group; (3) a fraction may represent the ratio of two quantities; and (4) a fraction represents an indicated division as in $2\overline{)5} = 2\frac{1}{2}$. The first and second of these concepts are best understood. Most students in the upper grades of the elementary school understand these meanings of fractions. All students in the junior high school have had experience with halves, quarters, and some of the other socially useful fractions. The ratio concept of a fraction is not so well understood as the first concept. The ratio concept of a fraction is essential in finding what per cent one number is of another number. Every fraction is an indicated division. Thus, if three apples are to be divided among four children, this situation can be represented by the fraction $\frac{3}{4}$. A student who does not understand that a fraction is an indicated division will not understand how to express a common fraction as an equivalent decimal fraction. To find the decimal equivalent of any common fraction, as $\frac{1}{2}$, the numerator is divided by the denominator. The numerator 1 of the fraction $\frac{1}{2}$ must be regrouped as 10 tenths, or 1.0, to complete the algorism. Further discussion of the topic will appear later in this chapter.

Differentiating the Curriculum

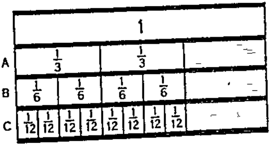
Some teachers are reluctant to accept a philosophy which sanctions differentiated subject matter in arithmetic for a given grade, such as the seventh grade. These teachers require the slow learners to deal with the same work in fractions as is required of students who are average or superior in ability in arithmetic. To do a satisfactory job in this subject in the advanced grades, the curriculum in fractions must be differentiated. This is true because of the great range in achievement in a typical class with different phases of fractions.[5] Differentiation of the curriculum in this phase of work can be achieved, first by differentiating the

[5] Brueckner, L. J. and Grossnickle, F. E. *Making Arithmetic Meaningful*, pp 88–92 Philadelphia: The John C. Winston Co., 1953.

subject matter and second, by differentiating the method of dealing with fractions. In most classes at the junior high school level, both of these means should be used to provide for differences in the students' backgrounds in fractions.

The slow learners should deal almost exclusively with those fractions which are used in socially significant situations. Fractions of this kind to be added or subtracted must represent parts of units of measurement. It is not possible to add ratios and have a sensible answer. It would not make sense to add the fractions $\frac{1}{2}$ and $\frac{3}{4}$ if each fraction represents the ratio of the lengths of two pencils, as pencil A is half as long as pencil B and pencil C is $\frac{3}{4}$ as long as pencil D. Most of our units of measures and weights are divided binarily, or to the base 2 or a power of 2, as shown by the number of subdivisions of an inch, a gallon, or a pound. Therefore, most social applications in addition and subtraction of common fractions will involve fractions having denominators of 2 or a power of 2, as 4, 8, and 16. If the fractions to be added or subtracted have denominators of this kind, the largest denominator always can be used as a common denominator for unlike fractions. Selecting the common denominator in this way should help the slow learners to add or subtract fractions of this type with little difficulty. On the other hand, students who show satisfactory achievement in the topic should be able to master the work in addition and subtraction of common fractions as found in a modern textbook in arithmetic.

The second way to differentiate the curriculum in fractions is to vary the method of presentation among groups of different levels of ability. The slow learner may find it necessary to use exploratory or visual materials to find answers to examples and problems involving fractions. The superior student should work chiefly with symbolic materials. He, too, should discover many mathematical relationships among fractions which the slow learner would not be able to discover or understand. Therefore, at the junior high school level, the teacher of a heterogeneous class of 30 or more students should form at least two or three groups within the class of average ability when dealing with common and decimal fractions. The students should be grouped according to their achievement and understanding of fractions



as measured by a standard achievement test in the subject, by tests found in good textbooks, or by tests made by the teacher. The students in the groups will not be homogeneous, but at least the range of backgrounds in fractions in each group will not be as great as the range of backgrounds in fractions of the whole class. As more groups are formed on the basis of background in fractions, the more homogeneous each group will become in that particular trait.

Illustration of Difference in Procedure among Groups

An illustration of the differentiation in method of dealing with different groups is shown in teaching the meaning of reduction of a fraction to lower terms, or of changing a fraction to higher terms. All students should understand that reducing a fraction to lower terms means to make the size of the equal parts larger. This makes the number of parts smaller as shown in the fractions $\frac{4}{6} = \frac{2}{3}$. The reverse is true in changing a fraction to higher terms. In this case, it means that the size of the equal parts is made smaller. This makes the number of equal parts larger. The teacher should use a fractional chart as shown to enable a student to make these generalizations. The use of cut-outs also should enable him to discover the relationship between the number of parts into which a whole is divided and the size of these parts. Then each group of students should see that a fraction may be reduced by dividing both numerator and denominator by the same number without changing the value of the fraction. Similarly, both terms of a fraction may be multiplied by the same number without changing the value of the fraction.

Most students at the junior high school level understand these two basic generalizations. Those students who do not understand these principles should use objective and visual materials for discovering these generalizations. The superior student should also understand why it is possible to multiply or divide both terms of a fraction by the same number without changing the value of the fraction. He should understand that multiplying or dividing each term of a fraction by the same number, as 3 in the illustration, is the same as multiplying or dividing the fraction by 1. The student should have learned that when a number is either multiplied or divided by 1, the answer is equal to that number. On the other hand, when 1 is added to a number or subtracted from a number, the answer has a different value from that number. The same number may not be added to or subtracted from both terms of a fraction without changing the value of that fraction. The student can verify the principle as shown by the examples on the right. The symbol, $\neq$, means "not equal." Few students in junior high school, or in higher grades, understand that the same number may not be added to or subtracted from both terms of a fraction without changing the value of the fraction. The reason is not due to the difficulty of the mathematical principle involved. Very probably the teacher had never given the student the opportunity to discover the relationship between operations with integers and fractions.

$$\frac{3 \times 3}{3 \times 4} = \frac{9}{12}$$

$$\frac{9 \div 3}{12 \div 3} = \frac{3}{4}$$

$$\frac{5}{6} = \frac{5}{6}$$

$$\frac{5+1}{6+1} \neq \frac{5}{6}$$

$$\frac{5-1}{6-1} \neq \frac{5}{6}$$

b. Teaching the Basic Operations with Common Fractions

Classification of Fractions in Addition and Subtraction

Fractions to be added or subtracted may be classified according to their denominators as follows:

A. Like fractions, as $\frac{1}{8} + \frac{3}{8}$

B. Unlike but similar fractions in which the largest denominator is a common denominator, as $\frac{1}{2} + \frac{1}{4}$

C. Unlike and dissimilar fractions
   (1) The product of the denominators is the lowest common denominator, as $\frac{1}{2} + \frac{1}{3}$
   (2) The product of the denominators is not the lowest common denominator, as $\frac{1}{4} + \frac{1}{6}$.

From the standpoint of social significance, the first and second types are most useful as was shown earlier in this chapter. Fractions to be added or subtracted usually represent a part of some unit of measurement. A measurement may be expressed as the nearest half, fourth, eighth, or sixteenth of a given unit, such as an inch or a pound. Then the fractions of this unit to be added or subtracted will all have denominators of 16, or factors of 16, such as 8, 4, or 2. Therefore, the fractions will have either like denominators, or if unlike, one of the denominators will be a common denominator. The slow learners should be able to add or subtract in examples in which the fractions are representative of the first and second classifications. The other group of students should be able to add or subtract in all types of examples in which the fractions are representative of all three classifications. The slow learners should use their fractional cut-outs and visual materials to find answers in addition and subtraction of fractions. The fast learners should have an enriched background in fractions which will enable them to deal predominantly with symbols.

Different Procedures for Slow and Fast Learners

The differences between the procedures to follow in the two groups can be illustrated by adding the fractions $\frac{3}{4}$ and $\frac{5}{8}$. The slow learners should use their cut-outs to find the sum. Then these students should identify the steps in a visual representation. Finally, these students should give the solution with symbols. However, the students who have a rich background of understanding of fractions should be able to give at once the solution with symbols as shown on the right and then check the work.

$$\begin{array}{rl} \frac{3}{4} = & \frac{6}{8} \\ +\,\frac{5}{8} = & \frac{5}{8} \\ \hline & \frac{11}{8} = 1\frac{3}{8} \end{array}$$

The student who develops the ability to think quantitatively with fractions is able to make a quick approximation of a given fraction with the fraction $\frac{1}{2}$. In the given example, he sees that each fraction is greater than a half, hence the sum of the fractions must be greater than a whole. There are only two proper fractions to be added; therefore, their sum must be less than two wholes. The sum, $1\frac{3}{8}$, is more than 1 and less than 2; hence the answer is sensible.

The superior student also should be able to make a sensible problem involving the two fractions. The term fraction is used to include mixed numbers as well as proper fractions. He should be able to make a problem involving fractional parts of pounds, inches, miles, or gallons. The student who is unable to make problems involving social applications of the addition of fractions has not reached the stage of mastery which the superior student should attain at the junior high school level.

Subtraction of Mixed Numbers

There is no apparent reason why a student should not be able to subtract in examples of the type shown on the right when he knows how to add unlike but similar fractions. He

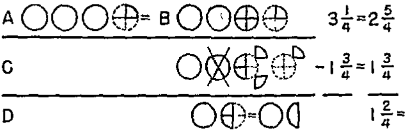
$$\begin{array}{r} \frac{1}{2} \\ -\ \frac{1}{4} \\ \hline \end{array} \qquad \begin{array}{r} 7\frac{2}{3} \\ -\ 1\frac{1}{6} \\ \hline \end{array}$$

would not discover the algorism for subtracting in examples with mixed numbers in which regrouping is needed without having instruction in the procedure to follow. The examples on the right represent the kind of examples in which the minuend must be regrouped in order to complete the subtraction. The slow learner should use objective and visual materials to find the answers to these examples. Eventually, he should reach the level of abstraction in dealing with quantities in which he operates with symbols.

$$\begin{array}{r} 6 \\ -\ 2\frac{1}{4} \\ \hline \end{array} \qquad \begin{array}{r} 3\frac{1}{4} \\ -\ 1\frac{3}{4} \\ \hline \end{array}$$

The method to use with different groups can be illustrated in the solution of the following problem:

A piece of cloth $1\frac{3}{4}$ yards long is cut from a piece $3\frac{1}{4}$ yards long. What is the length of the piece remaining?



First, the student should solve the problem by using his cutouts, by use of a ruler, or by some other objective means. Next he should identify the steps in a visual representation. Line A represents $3\frac{1}{4}$ and B shows this number regrouped as $2\frac{5}{4}$. Line C shows $1\frac{3}{4}$ subtracted from $2\frac{5}{4}$ and D shows the remainder of $1\frac{2}{4}$, or $1\frac{1}{2}$. The fast learner should give the symbolic solution shown above. He should be able to tell why the answer is sensible. His thought pattern might be as follows: "If 2 were subtracted from $3\frac{1}{4}$, the answer would be $1\frac{1}{4}$, but the remainder to the example must be more than $1\frac{1}{4}$. If 1 were subtracted from 3, the answer would be 2, but the remainder to the example must be less than 2."

The student who has developed insight into number might be able to give other thought patterns for checking the example. Finally, he should be able to make one or more similar problems involving the social application of these numbers. A problem could involve a unit of distance, weight, time, or capacity.

Unlike and Dissimilar Fractions

The social applications involving addition or subtraction of fractions which are dissimilar are very limited. It is for this reason that, for the slow learners, the curriculum in fractions in these two processes should be confined to like or similar fractions. The denominators of two dissimilar fractions may be *prime* with respect to each other, or there may be a common factor of the two denominators. Two numbers are prime with respect to each other if 1 is the only integer which will divide both numbers without a remainder. The denominators of the fractions $\frac{1}{2}$ and $\frac{1}{3}$ are prime to each other, as are any two consecutive numbers.

The denominators of the fractions $\frac{1}{4}$ and $\frac{1}{6}$ have a common factor of 2. The lowest common denominator (L.C.D.) of fractions having denominators which are prime to each other is the product of the denominators. The L.C.D. for the fractions $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{5}$ is 30, the product of the denominators. When there is a common factor of two or more denominators, their product will not be the lowest common denominator of the fractions. The product of the denominators will always be a common denominator, but this product will be the L.C.D. only when the numbers multiplied are prime with respect to each other. A common denominator of the fractions $\frac{1}{4}$ and $\frac{1}{6}$ is 24, but the L.C.D. is 12, or the product divided by the common factor.

In dealing with addition and subtraction of dissimilar fractions, the teacher should keep in mind two things about fractions of this kind. First, these fractions have limited social applications and are not part of the minimum program for slow learners. Second, the mathematical phase of the process should be stressed. The student should use approximation to determine whether or not an answer is sensible.

The teacher should use a minimum amount of objective and visual materials in dealing with examples involving adding or subtracting dissimilar fractions at the junior high school level. The student should have learned in the intermediate grades how to add two fractions of the kind shown on the right. He should have been taught always to see if the larger denominator of two unlike fractions is a common denominator. If the denominators are related, as in the fractions $\frac{1}{4}$ and $\frac{3}{8}$, the larger denominator is a common denominator. If the fractions are dissimilar, the larger denominator is not a common denominator.

$$\begin{array}{r} \frac{1}{2} \\ +\frac{1}{3} \\ \hline \end{array}$$

In an exercise of the kind which follows, the examples in which the L.C.D. will be the product of the denominators should be selected. Then the student should give the L.C.D. for the other examples.

a. $\frac{1}{3}$, $\frac{1}{4}$ b. $\frac{1}{2}$, $\frac{1}{5}$ c. $\frac{1}{6}$, $\frac{1}{9}$ d. $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$ e. $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{12}$

The student should discover that the L.C.D. in examples a, b,

and d will be the product of the denominators of each example, respectively because 1 is the only common factor of the numbers in each group. In example c, 3 is the highest factor of each number, therefore, the product of the denominators, 54, will be 3 times as large as the L.C.D., or 18. In e, the product of 8 and 12, or 96, will be 4 times as large as the L.C.D., or 24.

Adding Three or More Fractions

A	B
$\frac{1}{8}$	$\frac{2}{3}$
$\frac{3}{4}$	$\frac{5}{6}$
$\frac{9}{16}$	$\frac{3}{4}$

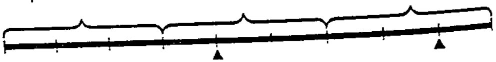
When an example contains three or more fractions to be added, the student should select the largest denominator to see whether it is a common denominator. In example A, the largest denominator is a common denominator, but in B, the largest denominator is not a common denominator. In this case, the student multiplies the largest denominator by 2, by 3, by 4, and so on until a common denominator is found.

Multiplication of Fractions

The most widely used form of multiplication of fractions is finding a fractional part of a number, as $\frac{1}{3}$ of 12. The student has learned that division has two uses. One of them is to find the size of equal parts into which a number is divided. This usage is the same as finding a fractional part of a number, as $\frac{1}{3}$ of 15. Therefore, finding the value of a unit fractional part of a number by multiplication involves the same process as dividing the number by the denominator of that fraction. The value of $\frac{1}{5}$ of 20 may be found by division, as $5\overline{)20}$, or by multiplication, as $\frac{1}{5} \times 20$. Thus, it is seen that multiplying by a number is the same as dividing by the reciprocal of the multiplier. Inversely, dividing by a number is the same as multiplying by the reciprocal of the divisor. Thus, $4 \div 3 = 4 \times \frac{1}{3}$.

Although finding a fractional part of a number is the most widely used form of multiplication of fractions, the algorism is not easy to understand. The easiest step to understand in multiplication of fractions consists in multiplying a fraction by a whole

number. In the example, $3 \times \frac{3}{4}$, it is possible to find the answer by use of objective materials or by addition. The slow learner should use his cut-outs to discover that three $\frac{3}{4}$'s are equal to $\frac{9}{4}$, or $2\frac{1}{4}$. Then he learns that the same answer can be found by addition, as $\frac{3}{4} + \frac{3}{4} + \frac{3}{4} = \frac{9}{4}$, or $2\frac{1}{4}$, or by multiplication, as $\frac{3 \times 3}{4} = \frac{9}{4}$, or $2\frac{1}{4}$.



The fast learner should multiply the given numbers and show that the answer is sensible. He should know that the answer must be less than 3 (3 ones) and greater than $\frac{3}{2}$ or $1\frac{1}{2}$, since $\frac{3}{4}$ is less than 1 and greater than $\frac{1}{2}$.

The slow learner first should use disks or other objective materials to find a fractional part of a number, as $\frac{1}{3}$ of 12. He should find the answer by use of objective materials and discover that the same result can be found by multiplying, as $\frac{1}{3}$ of $12 = \frac{1 \times 12}{3} = \frac{12}{3}$ or 4. In the same way he should find $\frac{2}{3}$ of 12 and then discover that finding any fractional part of a number may be solved by expressing the example in the form for multiplication of fractions, as $\frac{3}{4}$ of $15 = \frac{3 \times 15}{4}$. He should understand that the answer to the given example must be less than 15 because the multiplier is less than 1.

Mathematical Principles Involved in Multiplying by a Fraction

At the junior high school level, the superior student in arithmetic should be led to discover the mathematical principles involved in multiplying by a fraction. He should know that multiplication is a shortened form of addition of like numbers; therefore, he should understand the algorism for multiplying a fraction by a whole number, as $4 \times \frac{2}{3}$. Since the order in which two numbers are multiplied does not affect the product, the student should understand how to use multiplication to find a

fractional part of a number, as $\frac{1}{4}$ of 12 or $\frac{3}{4}$ of 7. The two mathematical principles involved in understanding the relationship between $3 \times \frac{1}{4}$ and $\frac{1}{4}$ of 3 and between $\frac{1}{4} \times 12$ and $4\overline{)12}$ are:

1. The order in which numbers are multiplied does not affect the product. Thus, $3 \times \frac{1}{4}$ and $\frac{1}{4} \times 3$ will have the same product. The notation for the process is more easily understood if written in the form $\frac{3 \times 1}{4}$ or $\frac{1 \times 3}{4}$.

2. Dividing by a number is the same as multiplying by the reciprocal of that number. Thus, dividing a number by 3 is the same as multiplying that number by $\frac{1}{3}$.

3. The order in which the multiplication and division operations are performed to find a fractional part of a number does not affect the product. The order of the processes is reversed in the two examples shown. In A, 3 times the number (20) is divided into 4 equal parts. In B, one-fourth of the value of the number is found by division and then this quotient is multiplied by 3. Example B illustrates the principle that an indicated product can be divided by a number by dividing only one factor by that number.

A.

$$\frac{3 \times 20}{4} = \frac{60}{4} = 15$$

B.

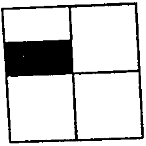
$$\frac{3 \times \overset{5}{\cancel{20}}}{\underset{1}{\cancel{4}}} = 15$$

Multiplication of a Fraction by a Fraction

The student learned in multiplication of a fraction by an integer, as $3 \times \frac{2}{5}$, that the notation of the example could be written as shown in A. The notation in B may also be used since the integer 3 means 3 ones which can be written as $\frac{3}{1}$. Then the product found by multiplying the numerators, 3 and 2, will be the numerator of the fraction in the answer, and similarly the product found by multiplying the denominators, 1 and 5, will be the denominator in the answer.

A. $\frac{3 \times 2}{5} =$

B. $\frac{3 \times 2}{1 \times 5} =$



The diagram shows that $\frac{1}{2}$ of $\frac{1}{4}$ is $\frac{1}{8}$. From a visual representation of the kind shown, the student should know that the answer of the example is $\frac{1}{8}$. Now he should find the answer by multiplication. Since he has already learned to multiply in examples of the type, $\frac{3 \times 4}{1 \times 5}$, he should experience no difficulty in multiplying two fractions of the type $\frac{1}{2} \times \frac{1}{4}$ or $\frac{2}{3} \times \frac{7}{8}$.

The student should understand why the product of two proper fractions, such as $\frac{1}{2} \times \frac{3}{4}$, is less than either fraction. He should understand that 1 times a number is equal to that number; therefore, $1 \times \frac{3}{4}$ will be $\frac{3}{4}$. When $\frac{3}{4}$ is to be multiplied by a number less than 1, the product will be less than $\frac{3}{4}$. Similarly, interchanging the fractions to be multiplied will give a product less than the other fraction, or $\frac{1}{2}$. The fast learners in arithmetic should be able to make the following generalizations:

1. The product of two proper fractions is less than either fraction.
2. The product of an improper fraction having a value greater than 1 and a proper fraction will be greater than the proper fraction, but less than the improper fraction.
3. The product of two improper fractions, each greater than 1, will be greater than either fraction.

Multiplying Three Fractions

The student in the upper grades may need to multiply three or more fractions when finding the volume of a prism. He should understand that the same procedure is followed for multiplying three fractions as applies to multiplying two fractions. Mixed numbers having a value less than 10 should be

changed to improper fractions. In the example, $2 \times 4 \times 5$, it is possible to multiply any two numbers, as 2 and 4, and then multiply that product by 5, but it is mathematically incorrect to multiply both 2 and 4 by 5 without changing the value of the example. Principle 3 given on page 144 governs the procedure to follow for multiplying a number of factors by another factor.

The student should discover that the order in which fractions are multiplied does not affect the product. He learned this principle when he applied it to the multiplication of whole numbers. Now he should see that it also applies to common fractions. In the example, $\frac{1}{2} \times \frac{3}{4} \times \frac{5}{8}$, it is possible to multiply $\frac{1}{2}$ and $\frac{5}{8}$ and that product by $\frac{3}{4}$, or the fractions may be multiplied in any other order. The usual way to write fractions to be multiplied is represented by the notation shown. In an exercise containing examples similar to those given below, the student should discover that the order of grouping some of the numbers in an example would simplify the computation.

$$\frac{1 \times 3 \times 5}{2 \times 4 \times 8} =$$

a. $\frac{3}{5} \times \frac{7}{8} \times 5$ b. $9 \times \frac{3}{4} \times \frac{7}{9}$ c. $\frac{3}{4} \times 5 \times \frac{7}{10}$ d. $\frac{1}{2} \times 9 \times \frac{3}{4}$

In a and b, the first and third numbers should be multiplied first; then in c, the second and third numbers should be multiplied first; in d, the order in which the numbers are multiplied will not affect the ease in computation. The function of an exercise of this kind is twofold: first, to enable the student to see that the principles which he learned about the order of multiplication of integers also apply to fractions; and second, to provide opportunities for him to discover relationships among quantities. It should be understood that an exercise of this kind is provided only for the fast learners in arithmetic.

Cancellation

Cancellation should not be taught as a new topic or process. In A, the fractions are multiplied and then the fraction in the answer is reduced to

A

$$\frac{3}{4} \times \frac{5}{6} = \frac{15 \div 3}{24 \div 3} = \frac{5}{8}$$

lowest terms by dividing both of its terms by 3. In B, numerator 3 and denominator 6 are divided by 3 before the fractions are multiplied. Cancellation is used in this example.

B

$$\overset{1}{\cancel{\tfrac{3}{4}}} \times \underset{2}{\tfrac{5}{\cancel{6}}} = \frac{1 \times 5}{4 \times 2} = \tfrac{5}{8}$$

Cancellation is based on the following principle: The product of two fractions will be the same if the terms of the fractions are multiplied and then both are divided by the same number, or if the division is performed before the fractions are multiplied. In examples of this kind, the teacher should emphasize the principle governing division of an indicated product. This principle states that to divide an indicated product by a number, only one factor of the product is divided by that number. In example B above, only one factor in each indicated product is divided by 3. In effect cancellation is a term used to represent the principle of reduction of the product in multiplication of fractions. The teacher should have the student use the term *division* to designate the operation. Often cancellation is a meaningless mechanical operation which is a trick to use for simplifying computation with fractions.

Types of Examples in Division of Fractions

The three types of examples in division of fractions are:

1. Dividing a whole number by a fraction
2. Dividing a fraction by a fraction
3. Dividing a fraction by a whole number.

1. Dividing a Whole Number by a Fraction

Division by a fractional divisor has limited social applications. For this reason it is doubtful if this topic should be included in a minimum program in arithmetic for all students in the junior high school.

The student who is able to understand the work in dividing by a fraction should have a variety of meaningful experiences to help him master the process. The student readily understands how to divide by a unit fraction. If a problem calls for finding

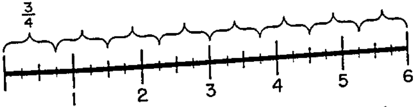
how many halves there are in 3 apples, the student should know that there are two halves in 1 apple and in 3 apples there will be 6 halves. This same answer can be found by inverting the divisor in the example, $3 \div \frac{1}{2} = 3 \times \frac{2}{1} = 6$, and multiplying to find the answer as shown.

If the divisor is not a unit fraction, as $\frac{2}{3}$, the student needs many different experiences to help him find the answer to a given problem. If a problem calls for finding the number of $\frac{3}{4}$-yard pieces that can be cut from 6 yards, the following procedures should be followed:

1. The student should measure the number of $\frac{3}{4}$ yards there are in a string 6 yards long.

2. He should use cut-outs or other objective materials to find the answer.

3. He should use a drawing similar to that given, to find the answer.



4. He should discover that there would be 24 pieces $\frac{1}{4}$ yard long. Since each piece is to be 3 times $\frac{1}{4}$ yard, there would be only one-third as many $\frac{3}{4}$-yard pieces as quarter-yard pieces, or 8 pieces in this case.

5. He should see the relationship between multiplication and division of fractions. He knows that $\frac{3}{4} \times 8 = 6$, then $6 \div \frac{3}{4}$ must be 8.

6. The student should learn that a number may be divided by multiplying it by the reciprocal of the divisor. The reciprocal of a fraction is the fraction inverted, hence, 6 may be divided by $\frac{3}{4}$ by multiplying by $\frac{4}{3}$.

7. He should understand why the answer is sensible. If 6 were divided by 1, the quotient would be 6. Since the divisor is less than 1, the quotient must be greater than 6. The student must

understand that division in the example, $6 \div \frac{3}{4}$, means finding how many times as large 6 is as $\frac{3}{4}$. The answer can be found either by subtraction or by division.

When he measured the number of $\frac{3}{4}$-yard lengths in a string 6 yards long, he subtracted $\frac{3}{4}$ yard from 6 yards 8 times. If a student experiences the seven different ways given for dividing a whole number by a fraction, he should have a satisfactory understanding of the process. Many students at the junior high school level will not understand the mathematical reason for inverting the divisor and multiplying.

Why We Invert the Divisor and Multiply

The example, $6 \div \frac{3}{4}$, can be written in the more familiar division notation as, $\frac{3}{4}\overline{)6}$. If the divisor, $\frac{3}{4}$, were 1, the solution would be easy. The product of a fraction and the fraction inverted is 1. Thus, $\frac{3}{4} \times \frac{4}{3} = 1$. In the example, $\frac{3}{4}\overline{)6}$, if $\frac{3}{4}$ is multiplied by $\frac{4}{3}$ to give a product of 1, then 6 also must be multiplied by $\frac{4}{3}$ to keep the value of the example the same. The example may be written as shown:

$$\frac{4}{3} \times \frac{3}{4}\overline{)\frac{4}{3} \times 6} = 1\overline{)8}, \text{ or } 8.$$

Since multiplying a fractional divisor by its reciprocal always gives a product of 1, the work may be shortened by multiplying the dividend by the reciprocal of the divisor. From this fact we derive the rule that *to divide by a fraction, invert the divisor and multiply*. Duker suggested that a meaningful and understandable rule for the learner may be stated as follows: "In order to divide by a fraction we multiply the dividend by the reciprocal of the divisor."[6]

The mathematical principles involved in inverting a fractional divisor and multiplying in a division example are:

1. The product of a fraction and the fraction inverted is 1.
2. Both divisor and dividend may be multiplied by the same number without changing the value of the example.

[6] Duker, S. "Rationalizing Division of Fractions," *The Arithmetic Teacher*. 1:22.

3. Inverting a fractional divisor and multiplying by it makes the effective divisor 1.

A Fractional Divisor

A fractional divisor may be either a proper fraction or an improper fraction. The dividend may be an integer, a proper fraction, or a mixed number. The student should first learn to divide an integer by a proper fraction or a mixed number before learning to divide a fraction by a fraction.

There are several ways to divide by a common fraction. The two most familiar methods are (1) by inverting the divisor and then multiplying and (2) by expressing both divisor and dividend as fractions having the same denominator and then dividing the numerators. Grossnickle[7] has designated the first method the *inversion method* and the second, the *common denominator method.* The two methods may be illustrated in the example, $6 \div \frac{3}{4}$:

$$(1)\quad 6 \div \tfrac{3}{4} = \tfrac{6}{1} \times \tfrac{4}{3} = \frac{6 \times 4}{1 \times 3} = \tfrac{24}{3},\text{ or } 8$$

$$(2)\quad 6 \div \tfrac{3}{4} = \tfrac{24}{4} \div \tfrac{3}{4} = \frac{24 \div 3}{4 \div 4} = \tfrac{8}{1},\text{ or } 8.$$

The preponderance of usage of the inversion method justifies the teaching of this method. The most objectionable feature of this method may be the pupil's lack of understanding of the process. Frequently, pupils invert the divisor and multiply without understanding the operation. Since most teachers accept the principle of meaningful learning, it is necessary to define meaning as applied to division by a fractional divisor.

Brownell[8] suggested that there are three phases of meaning in arithmetic, as quoted on the next page.

[7] Grossnickle, Foster E. "How to Use a Fractional Divisor," *Journal of Education.* 137:17–19.

[8] Brownell, William A. "When Is Arithmetic Meaningful?" *Journal of Educational Research.* 38:484.

The essential meanings of arithmetic, apart from such uses as involved, for example, in mensuration are:

1. Meanings of whole numbers, common fractions, decimals, per cents, together with understanding of the number system and of place values.
2. Understanding of the functions of the basic operations of addition, subtraction, multiplication, and division.
3. Understanding of the rationale of computations,—the forms used, the placement of the partial products, of quotient figures, and the like.

The writers agree with Brownell on items 1 and 2. Their views are not in complete agreement with item 3. They would modify item 3 to agree with their views in regard to meaning in arithmetic. A pupil need not understand the basis of all of the sequential steps in a process, such as division by a two-place number or by a fractional divisor.

In such a difficult process as division by a fractional divisor, meaning implies that a student should be able to state whether or not an answer is sensible. Many students in Grades 6 and 7 will profit little from instruction aimed at teaching the rationale of a difficult process, such as the use of a fractional divisor by the inversion method. It is possible, however, to teach the student to discover a method of finding the answer and to decide if the answer is reasonable. In the example, $6 \div \frac{2}{3}$, the quotient is 9. The student should know that this answer is sensible because he can discover the answer through use of exploratory materials. Also 6 divided by $\frac{1}{2}$ is 12 and 6 divided by 1 is 6. Since the divisor, $\frac{2}{3}$, is greater than $\frac{1}{2}$ but less than 1, the quotient must be less than 12 but greater than 6. The answer 9 is within this range; therefore, 9 is a sensible answer.

In the example, $\frac{2}{3} \div \frac{3}{4} = \frac{8}{9}$, the quotient is sensible because a smaller number is divided by a larger number, hence the quotient must be less than 1. When a fraction is divided by a fraction, the answer can be checked to see if it is sensible by the following three generalizations which a student in the junior high school should understand:

1. When a number is divided by itself the quotient is 1.
2. When a larger number is divided by a smaller number, the quotient is greater than 1.

3. When a smaller number is divided by a larger number, the quotient is less than 1.

A fourth generalization applies in particular to dividing an integer or a mixed number by a proper fraction. In an example of this kind, the quotient is greater than the dividend because the divisor is less than 1. The student should discover this principle from meaningful experiences of the kind shown on page 193.

Checking to See if the Answer Is Sensible

The student has the essential understanding for determining whether an answer is sensible in division of fractions if he is able to make the following generalizations about the answers in the given examples:

a. $\frac{1}{4} \div \frac{7}{8}$	a. The quotient must be less than 1. (Principle 3)
b. $3 \div \frac{5}{6}$	b. The quotient must be greater than 3. (Principle 4)
c. $\frac{5}{6} \div 1\frac{3}{4}$	c. The quotient must be less than 1. (Principle 3)
d. $2\frac{1}{2} \div 1\frac{2}{3}$	d. The quotient must be more than 1. (Principle 2)

In each example above, the student should invert the divisor, frequently a mechanical process, and complete the solution. Then he should check to see if the answer is sensible. If the answer does not agree with a sensible check, he should look for the source of error.

Meaning in arithmetic does not imply "all rationalization and no mechanization," but instead, it implies that there should be a minimum of mechanical operations. When part of a difficult algorism is performed mechanically, such as inverting the divisor and multiplying, a rational check must be applied. After the use of that mechanistic procedure it is important to ascertain that the answer is sensible. An understanding of these basic principles of division provides the basis for such a rational check.

The Common Denominator Method

Some teachers use the common denominator method for dividing by a fraction because they assume that division by this method is easier for the student to understand than division by the inversion method. In the program of Developmental Arithmetic in New York City, the common denominator method was tried recently on a wide experimental basis. The results proved that the plan created so much confusion in the student's mind that the method was completely abandoned. When a smaller fraction was divided by a larger fraction, as shown, often the student did not know if he should divide 2 by 3 or 3 by 2. To

$$\tfrac{1}{3} \div \tfrac{1}{2} = \tfrac{2}{6} \div \tfrac{3}{6} = \frac{2 \div 3}{6 \div 6} = \frac{\tfrac{2}{3}}{1} = \tfrac{2}{3}$$

make this method effective, the student should be able to estimate if the quotient is to be more than 1 or less than 1. The algorism of inverting the divisor and multiplying is easier for the student than finding the common denominator of the fractions and then dividing. In view of the fact that the common denominator method proved ineffective in the experimental program in New York City and that the method in any case should be supplemented by a sensible check on the answer, the inversion method which is the most widely used of all methods is recommended for dividing by a fraction.

2. *Dividing a Fraction by a Fraction*

There are few social applications of dividing a proper fraction by another proper fraction, especially dividing a smaller fraction by a large fraction, as $\frac{1}{3} \div \frac{1}{2}$. However, there are social applications of dividing an integer or a mixed number by a mixed number. Usually, the easiest way to divide mixed numbers is to change them to improper fractions, hence the student must know how to divide a fraction by a fraction.

The student should learn to divide by a fraction by inverting the fraction and multiplying. As pointed out before, it is difficult

to make this operation mathematically meaningful to many students in the junior high school. Therefore, emphasis is placed on knowing that an answer is sensible. The solution on the right shows how to divide a proper fraction by a proper fraction. To a great extent this is a mechanical operation. The student who understands the meaning of division of fractions should be able to discover whether or not $\frac{2}{3}$ is a sensible answer of the example, $\frac{1}{2} \div \frac{3}{4}$. The test for finding whether or not an answer is sensible in division of fractions consists in the application of the three generalizations given on pages 196–197.

$$\frac{1}{2} \div \frac{3}{4} = \frac{1}{2} \times \frac{4}{3} = \frac{4}{6}, \text{ or } \frac{2}{3}$$

3. *Dividing a Fraction by a Whole Number*

Of the three types of examples in division of fractions, dividing a fraction by an integer is the easiest kind for the student to understand. The answer seems sensible because it is smaller than the dividend. This also is true in division of integers except when the divisor is 1.

The slow learners use their cut-outs to find the answer to an example, as $\frac{1}{3} \div 2$. The student knows that to divide an object into two equal parts is the same as finding half of it. Therefore, dividing a number by 2 is the same as finding half of a number. Thus, $\frac{1}{3} \div 2$ is the same as multiplying $\frac{1}{3}$ by $\frac{1}{2}$ or $\frac{1}{2} \times \frac{1}{3}$. From illustrations of this kind the student should discover that a fraction may be divided by a whole number by multiplying by the reciprocal of the integer. The reciprocal of an integer is 1 divided by that number. Thus, the reciprocal of 4 is $\frac{1}{4}$.

The fast learner should understand that dividing a fraction by an integer follows the same pattern as dividing whole numbers. He learned that the partition concept of division consists in finding a fractional part of a number. Thus, finding $\frac{1}{4}$ of 12 is the same as dividing 12 by 4. Inversely, dividing 12 by 4 is the same as multiplying 12 by the reciprocal of the divisor, or $\frac{1}{4} \times 12$. (See principle No. 4 on page 144.) In the same way $\frac{2}{3}$ divided by 4 would be found by multiplying $\frac{2}{3}$ by $\frac{1}{4}$, or $\frac{1}{4} \times \frac{2}{3}$.

At the junior high school level, the student who develops insight into number should discover that a fraction may be divided by a whole number by two different procedures. Either the numerator of the fraction may be divided by the integer, as $\frac{4}{5} \div 2 = \frac{2}{5}$, or the denominator of the fraction may be multiplied by the integer, as $\frac{3}{4} \div 2 = \frac{3}{8}$. Since dividing a fraction by a whole number means to make the parts of the fraction smaller, every fraction can be divided by an integer by multiplying the denominator by the integer. Thus, $\frac{4}{5} \div 2 = \frac{4}{5 \times 2} = \frac{4}{10}$, or $\frac{2}{5}$.

Complex Fractions for Enrichment

The fast learner can have his understanding of fractions enriched by a meaningful treatment of *complex fractions*. The example, $\frac{3}{4} \div 2$, may be written in the fraction form as $\frac{\frac{3}{4}}{2}$. A complex fraction has a fraction in either the numerator or the denominator or in both. Fractions of this kind appear in algebra and in per cent as shown on page 201. The complex fraction shown has $\frac{3}{4}$ as its numerator. The student should have discovered from examples of the type, $3 \times \frac{2}{3}$, that, when a fraction is multiplied by a number equal to the denominator of the fraction, the product will be the numerator of the fraction. In the example $3 \times \frac{2}{3}$, the product is 2 which is the numerator of the fraction $\frac{2}{3}$. The student should know that both numerator and denominator of a fraction may be multiplied by the same number without changing the value of the fraction. This principle applies to complex fractions. The application of this principle makes it possible to simplify a complex fraction. In the example, $\frac{\frac{3}{4}}{2}$, if both terms of the complex fraction are multiplied by 4 (the denominator of the fraction, $\frac{3}{4}$), the complex fraction is changed to the equivalent fraction $\frac{3}{8}$. Therefore, $\frac{3}{4} \div 2 = \frac{3}{8}$.

The fast learner needs to know how to solve complex fractions when he expresses certain per cents as equivalent common fractions. One of the easiest ways to change a per cent to a common fraction is to drop the per cent symbol and divide the number

by 100. To illustrate, 25% $= \frac{25}{100}$, or $\frac{1}{4}$. Similarly, $12\frac{1}{2}\% = \frac{12\frac{1}{2}}{100}$. This complex fraction can be simplified by multiplying both of its terms by 2 as shown on the right. An example involving dividing a fraction by an integer, as $\frac{2}{3} \div 4$, should be solved by two methods.

$$\frac{2 \times 12\frac{1}{2}}{2 \times 100} = \frac{25}{200}, \text{ or } \frac{1}{8}$$

$$\frac{\frac{2}{3}}{4} = \frac{3 \times \frac{2}{3}}{3 \times 4} = \frac{2}{12}, \text{ or } \frac{1}{6}; \qquad \frac{2}{3} \div 4 = \frac{2}{3} \times \frac{1}{4} = \frac{2}{12}, \text{ or } \frac{1}{6}.$$

Only students benefiting from an enriched program in arithmetic, however, should deal with complex fractions.

Discovering Relationships among Operations with Fractions

On page 169 the teacher found the kind of exercise to give to fast learners in arithmetic so as to have them discover the relationship between addition and multiplication and their inverse processes, respectively, when dealing with integers. Exercises dealing with common fractions should be provided for similar purposes. The student should use the three fractions in box A and make two examples in addition and the corresponding examples in subtraction. One of the examples in addition is $\frac{1}{4} + \frac{1}{2} = \frac{3}{4}$. The teacher should not indicate the process to use. The student should discover if the first two numbers must be added or multiplied in order to give the third number.

A

$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$

Most students in the junior high school, who have a good background in multiplication of fractions, have little difficulty making two examples in multiplication from the numbers in box B. It is much more difficult for these students to make the corresponding examples in division In that case, the teacher should review how to make a division example with integers as shown in box C. The principle states that if the product of two numbers is divided by one number, the quotient will be the other

B

4	$\frac{2}{9}$	$\frac{8}{9}$

C

3	5	15

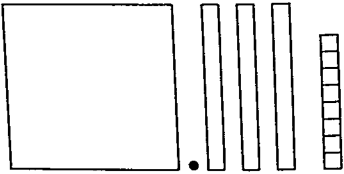
number. Using the numbers in B, if the product of the two numbers, $\frac{8}{9}$, is divided by one number, as 4, the quotient will be the other number, or $\frac{2}{9}$. Only the very superior student is able to discover how to write the corresponding example in division from an example in multiplication of fractions, as in the example, $\frac{2}{3} \times \frac{3}{4} = \frac{1}{2}$.

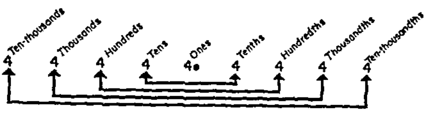
c. Teaching the Basic Operations with Decimal Fractions

Meaning of Decimal Fractions

Most of the students at the junior high school level understand the meaning of a decimal expressed as tenths or hundredths. The student who does not understand how a part of a unit can be expressed as a decimal fraction should use his squares as described on page 147. He should discover that the large blank square is equal to 10 equal strips, and that each strip is equal to one-tenth of the whole. One-tenth may be written as .1 or $\frac{1}{10}$. From the use of his kit materials he should discover that $1 = \frac{10}{10}$ or 1.0. Then the teacher should have a student demonstrate the same fact by use of markers in place-value pockets. In a similar manner the student should learn to express a unit as hundredths by using his kit materials and also markers in a place-value pocket. The diagram shows how to use the squares to represent the number 1.38.

When the student understands the meaning of tenths and hundredths from use of his objective materials and visual representations of these places, he should discover that the next subdivision of a decimal fraction would be thousandths. If the





student is unable to make this discovery, he should review the study of the meaning of tenths and hundredths by further use of objective and visual materials. The student who does not develop enough insight into the structure of our number system to discover the meaning of thousandths in relation to ones, tenths, and hundredths will derive little benefit from performing operations with decimals expressed in more than two places. His decimal usages most probably will be confined to their application with money.

The student should discover the relationship between the names of the places to the left of ones' place and the names of the corresponding places to the right of ones' place. The position of a place on the number scale should be determined from the ones' place and not from the decimal point. In the number 304.256, the 3 should be identified on the number scale as 2 places to the left of ones' place and not as 3 places to the left of the decimal point. The 6 is 3 places to the right of both ones' place and the decimal point, but the ones' place should be used as the point of reference from which to identify the position of a figure in a number. One place to the left of ones' place identifies tens and one place to the right of ones' place identifies tenths. Similarly, two places to the left represents hundreds and two places to the right represents hundredths. From the illustrations it is seen that the decimal point identifies ones' place. The point should not be used to indicate the place from which to identify the value of a position.

Place Value Understandings for Superior Students

An excellent type of problem for the superior student is one in which he compares the value of a digit expressed in different

positions in a number, as the 2's in 292, or in 2.02. The value of the 2 in hundreds' place in the number 292 is 100 times the value of the 2 in ones' place. The relationship is the same when comparing the values of the 2's in 2.02. The value of the 2 in ones' place is 100 times the value of the 2 in hundredths' place. There is more difficulty involved when the values of two different digits are compared, such as 2 and 4 in the numbers 24 or 4.02. In the junior high school, only the very superior students in arithmetic are able to express the relationship between the values of the digits 2 and 4 in an example of the type 2.14. Many students who can express the relationship between the values of the digits 2 and 4 in the number 214 cannot give the ratio of the two values in the number 2.14. This proves that the students do not have a clear concept of the role of the decimal point in a number.[9]

The point identifies the ones' place, but it does not affect the relationship between the values of any two digits in a number. Thus, the value of 1 in each of the numbers 1034, 10.34, and .1034 is 250 times the value of the 4 in these numbers.

The student should express the ratio of the digit having the greater value to the digit having the smaller value and then in the reverse order. In the number, 272, the value of the 2 in hundreds' place is 100 times the value of the 2 in ones' place. Therefore, the value of the 2 in ones' place is $\frac{1}{100}$ of the value of the 2 in hundreds' place. The student should discover that moving a digit to the left multiplies the digit by the power of 10 for each place the figure is moved. The relationship between multiplication and division applies in this situation. The student should have learned that multiplying by a number is the same as dividing by the reciprocal of the number. Therefore, if one digit in a number has a value 250 times the value of another digit, as is true of the digits 1 and 4 in the number 1254, the ratio of the values of these digits in the reverse order is $\frac{1}{250}$. The teacher should provide exercises of this kind for enrichment purposes for those who excel in number. These students develop deeper insight into number than others.

[9] See Johnson, T. J. "The Use of a Ruler in Teaching Place Value in Numbers," *The Mathematics Teacher*. 45:266.

The teacher should have the class make collections of uses of decimals found in newspapers and magazines, and in other places. Collections of this kind furnish excellent material for display on the bulletin board. To supplement current material, the teacher should have on file pertinent materials collected in past years. A student committee should be chosen to select materials to be used for display purposes and to arrange them effectively on the bulletin board.

Most boys at the junior high school level know the batting averages of the leading hitters in the major leagues, earned run averages of pitchers, rankings of teams, the average number of points scored per game of star basketball players, and other statistics found in sports. The student should understand that batting averages and team standings are expressed as three-place decimals, rounded off from a fourth place. In this way he learns that a decimal is rounded off in the same manner as a whole number. The teacher may use this as one more illustration to show that the decimal point does not affect the operation which may be performed with numbers.

A striking illustration of the use of decimals in baseball occurred in 1949. In that year it was necessary to express the batting average to four decimal places to determine the batting champion in the American League. The two contenders were Kell and Williams whose records were as follows:

Batter	*At Bat*	*Hits*	*Batting Average*
Kell	522	179	.3429
Williams	566	194	.3428

For the first time in the American League it was necessary to carry the batting average to four decimal places to determine the batting champion. From the given data it is seen that Kell won the batting championship by one ten-thousandth. The teacher should capitalize on the interest shown by most students in the use of decimals in sports. Often a slow learner can interpret decimals as used in sports, but he may be unable to understand

the mathematical relationships involved in decimal fractions. There are many American youths who know the significance of a "300 hitter," but they may never understand much about the mathematical interpretation to be given to the simple expression .300.

A list of current uses of decimals will show that frequently a group name without the zeros as place holders will be used to express a large number. The number, 4,500,000, can be expressed as 4.5 million. Of course, the number also can be expressed as 4500 thousand. When dealing with millions and billions, it is predominantly group names instead of zeros that are used to express numbers. At a recent date leading newspapers stated that our population was 167.4 million. The student must learn to interpret and to write this number with figures. The number, 167.4 million, means 167 million, or 167,000,000, and .4 million which is .4 of 1,000,000, or 400,000. Therefore, 167.4 million is the same as 167,400,000. In newspapers the group name of a large number is used predominantly in place of zeros because many readers are unable to distinguish between millions and billions when zeros are used as place holders in large numbers. It is much easier for the reader to get the correct concept of the meaning of the quantity \$7.5 billion when written in that form than as \$7,500,000,000.

Addition and Subtraction of Decimals

The ease with which computation may be made with decimal fractions represents one of the great contributions of the decimal number system. Buckingham described this fact succinctly as follows:

> In the handling of numbers the dream of the algorist was to free men from a machine. . . . For centuries the algorist's dream came true only in the domain of the natural numbers. At the frontier between whole numbers and fractions his conquest of arithmetical illiteracy ended.
>
> Then came the great sixteenth century and with it, among other things, the discovery of decimals. By this discovery the "front" between the domain of whole numbers and fractions was

> obliterated. The algorist had completed his work. His notation embraced the inconceivably large and the inconceivably small. The forms which he bequeathed to humanity held good throughout the realm of fractions as they had already been found to hold in the realm of integers.[19]

The student adds or subtracts decimals as he adds or subtracts integers. He has learned that only like quantities with integers can be added or subtracted. This principle applies to decimals and it means that tenths must be combined with tenths, hundredths with hundredths, and so on. The student should learn to approximate the answer in addition and subtraction of decimals as he did with whole numbers. If a student knows that only numbers in like places can be combined and knows how to approximate an answer, he should understand the work in addition and subtraction of decimals. It is assumed that the student already knows how to perform these processes skillfully with integers.

Ideally, the use of *ragged decimals* as shown in A should not be tolerated. Since numbers to be added or subtracted usually represent measurements of some kind, all measurements to be combined must be expressed to the same degree of precision. In the given example, all numbers must be expressed as tenths, hundredths, or thousandths. Theoretically, the numbers should be rounded off to the least precise measurement, namely, tenths. The example then would be written as shown in B. At the junior high school level, the student should be instructed to fill in the blank places so as to express each number to the same accuracy as the most precise measurement, namely, thousandths. The example containing the ragged decimals then would be written as shown in C. Many standard tests in arithmetic perpetuate the use of ragged decimals. These tests have helped to keep this obsolete material in many courses of study.

A	B	C
4.3	4.3	4.300
1.56	1.6	1.560
0.385	0.4	0.385

The student approximates the sum of an example which contains *pure decimals* (less than 1) as shown in

[19] Buckingham, B. R. *Elementary Arithmetic: Its Meaning and Practice*, pp. 344–45. Boston: Ginn and Company, 1947.

D, by rounding off the numbers to the nearest tenth or to the first place in which significant figures are written. The approximate sum would be 1.1. The true value of the sum is 1.200. If mixed decimals, as 42.5, are added, the student should round off the numbers to be added in the same manner as when dealing with integers. Then he should find the approximate sum.

$$\begin{array}{r} \text{D} \\ .483 \\ .129 \\ .543 \\ .045 \\ \hline 1.200 \end{array}$$

In the discussion of D above, it was suggested that the true value of the sum is 1.200 and not 1.2. Each of the numbers is expressed as thousandths, therefore the sum must be expressed as thousandths. If the sum is given as 1.2, the maximum error in measurement is 100 times as great as the maximum error in the number 1.200.

Multiplying a Decimal by an Integer

The student should have learned that the decimal point identifies ones' place, and that the point does not affect the method of computation within a process. Therefore, the algorism of multiplication of decimals is the same as multiplication of integers except for the identification of ones' place in the product which is the equivalent of the placement of the point in the product. The slow learner can use his kit material to objectify multiplying a decimal by a whole number, as in the examples on the right. He also can find the answer by addition and by use of common fractions. From varied experiences of this kind with objective and symbolic materials, he should understand how to multiply a decimal by an integer.

$$\begin{array}{r} \text{A} \\ .4 \\ \times 3 \\ \hline 1.2 \end{array} \qquad \begin{array}{r} \text{B} \\ 1.8 \\ \times 2 \\ \hline 3.6 \end{array}$$

Most of the members of a class at the junior high school level should be able to approximate the answer in multiplying a decimal by an integer to determine the position of the point in the product. Then the work can be checked by applying a rule, such as pointing off as many places in the product as there are decimal places in the number multiplied. In example A given above, the student should think, ".4 is a little less than a half or .5, so the product must be a little less than 1.5. The answer must be 1.2." In B, the thought pattern would be, "1.8 is a little

less than 2, so the product must be less than 4." In an example of the type shown on the right, the student should think, ".075 is a little less than .1, hence the product must be less than .5. The answer must be .375." When a student deals with decimals in this manner, he shows that he understands their value.

$$\begin{array}{r} .075 \\ \times 5 \\ \hline .375 \end{array}$$

Multiplying by a Power of 10

The student should discover that multiplying a decimal by 10 or any power of 10 is the same as moving the decimal point as many places to the right as the power of 10 indicates. At the beginning, the student should be permitted to write the work in the long form as shown. Then he should be able to generalize about the position of the decimal point in the product when the multiplier is a power of 10. The ability to multiply by 10, 100, or 1000 is needed in division of decimals when the divisor is a decimal. When a decimal divisor is changed to a whole number by multiplying the divisor by a power of 10, the dividend also must be multiplied by the same power of 10. Therefore the student must be able to multiply a decimal by 10, 100, or 1000 and he must know by what power of 10 to multiply to have the product a whole number. Thus, the student must know that $1000 \times .045 = 45$ and that it is necessary to multiply 3.14 by 100 in order to have the product equal to 314.

$$\begin{array}{r} 7.4 \\ \times 10 \\ \hline 74.0 \end{array}$$

Multiplying an Integer by a Decimal

At the junior high school level, all members of a class should be able to approximate the answer to a problem of the following type: How far will a car travel in 1.5 hours at an average speed of 45 miles per hour? The student learned that the decimal point identifies ones' place, hence the point does not affect the computation. In this example, he should see that the distance must be more than 45 miles and less than 90 miles; therefore, 67.5 miles must be the correct answer, not 675 or 6.75. From

$$\begin{array}{r} 45 \\ \times 1.5 \\ \hline 225 \\ 45 \\ \hline 67.5 \end{array}$$

similar illustrations the student should be led to discover that there are as many decimal places in the product as there are decimal places in the multiplier.

The student learned that interchanging the factors of a product does not affect the product. He also learned how to multiply a decimal by an integer. Therefore, he should be able to multiply $3 \times .5$ and from that he should know that $.5 \times 3$ would have the same product.

The student can check the given example, 45×1.5, by expressing 1.5 as its equivalent mixed number and then multiplying 45 by $1\frac{1}{2}$. A decimal fraction has an indicated denominator of a power of 10. Thus, the decimal fraction, .3, can be expressed as a common fraction as $\frac{3}{10}$. Similarly, the decimal fraction, .03, can be expressed as a common fraction, as $\frac{3}{100}$. By use of common fractions, the answer of the given example will prove that the product of 45 and 1.5 must be 67.5. Thus, all students at the junior high school level should find the position of the decimal point in the product of an integer multiplied by a decimal by three different means.

1. Apply the principle that interchanging the factors of a product does not affect the product.
2. Express the decimal as an equivalent common fraction and then multiply.
3. Estimate the approximate product, especially when the multiplier is a mixed decimal.

Multiplying a Decimal by a Decimal

Examples A, B, and C represent the three basic types of examples found in multiplying a decimal by a decimal. The product is easiest to estimate in type A. In A, the product must be slightly more than 4 (1×4), hence the only sensible answer is 5.04. The use of common fractions should enable the student to understand that the product of tenths and tenths is hundredths as shown in types

A	B	C
4.2	.6	.3
×1.2	×.3	×.2
5.04	.18	.06

B and C. From similar exercises the student should discover that *the number of decimal places in the product is equal to the sum of the number of decimal places in the numbers multiplied.* At the junior high school level, students who have a background for understanding multiplication of decimals should be able to discover the above principle.

The student who develops a high degree of insight into the meaning of the number system should be able to understand the value of a product in terms of the place value of its factors. This type of student should be able to generalize as follows:

1. The product of ones and ones is ones, as $3 \times 2 = 6$.
2. The product of ones and tenths is tenths, as $2 \times .4 = .8$.
3. The product of tens and tens is hundreds, as $20 \times 30 = 600$.
4. The product of tenths and tenths is hundredths, as $.4 \times .3 = .12$.
5. The product of ones and hundredths is hundredths, as $4 \times .03 = .12$.
6. The product of ones and the value of any place has the value of that place.

At the completion of the junior high school, the superior student should be able to understand the place value of products of the type shown. In the first principle given, the product of ones and ones is ones. This principle is easily understood in an example of the type, $2 \times 3 = 6$, but it may not be understood in an example of the type, $3 \times 6 = 18$. In the latter example, 6 ones are multiplied by 3 ones and the product is 18 ones which is an ungrouped number. This number may be regrouped as 1 ten and 8 ones. In a similar way other products may be regrouped. It should be understood that only the superior student will be able to profit from an exercise in which he locates the position of the point in the product from knowing the product of the place values of two factors, such as the product of tenths and tenths is hundredths. Material of this kind is for enrichment purposes to challenge the superior student so that he will grow in understanding of the number system and also of the principles which govern the operations within the number system.

Dividing a Decimal by an Integer

There are four types of examples in division of decimals. They are as follows:

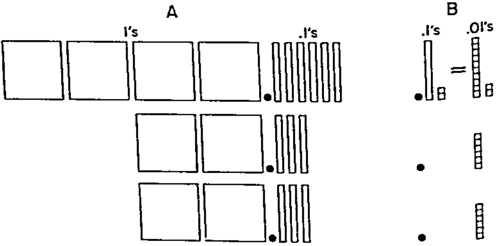
1. Dividing a decimal by an integer, as $2\overline{).6}$; $2\overline{)3.2}$
2. Dividing two whole numbers with a decimal in the quotient, as $5\overline{)2}$; $12\overline{)18}$
3. Dividing a whole number by a decimal, as $.5\overline{)3}$; $1.5\overline{)6}$
4. Dividing a decimal by a decimal, as $.5\overline{).12}$; $3.6\overline{)4.68}$

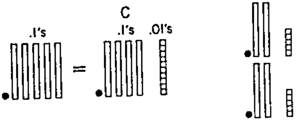
In types 1 and 2, the divisor is a whole number and in types 3 and 4 the divisor is a decimal. Type 1 is easiest for the student to understand.

Examples X, Y, and Z represent the three basic types of examples found in dividing a decimal by a whole number. In Y, the 1 tenth must be regrouped as 10 hundredths, making a total of 12 hundredths. In Z, 5 tenths must be regrouped as 4 tenths and 10 hundredths in order to complete the necessary division. Diagrams A, B, and C show how the student should use his decimal squares to find the answer to examples X, Y, and Z.

X	Y	Z
$\begin{array}{r} 2.3 \\ 2\overline{)4.6} \end{array}$	$\begin{array}{r} .06 \\ 2\overline{).12} \end{array}$	$\begin{array}{r} .25 \\ 2\overline{).5} \end{array}$

The first row in each diagram shows the number to be divided. In diagrams B and C, the numbers represented must be regrouped. The second and third rows show the number in the first row divided into two equal parts.



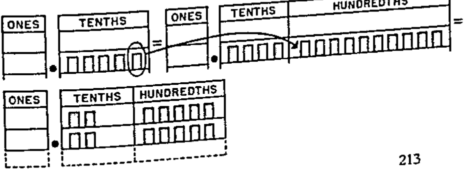


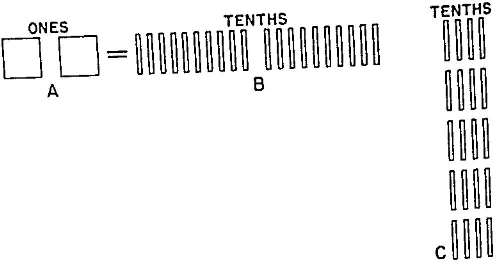
The teacher should make a visual representation of an example on the chalkboard as shown below Each student should be able to tell the steps involved in the representation. Here we have shown .5 divided into two equal parts.

Next, the student should make a symbolic representation of the examples which he solved with his squares. The slow learners will have to use objective materials for a longer time than the fast learners. As soon as a student is able to approximate an answer, he should discontinue the use of objective and visual materials. In the example, $3\overline{)5.4}$, the student who has an understanding of the process should be able to state that the quotient will be slightly less than 2.

Dividing Two Integers with a Decimal Quotient

There are two basic types of example which characterize dividing an integer by another integer with a decimal in the quotient. They are: (1) expressing a common fraction as a decimal fraction, as $\frac{1}{2} = .5$; and (2) expressing the ratio of a larger number to a smaller number, as $\frac{3}{2} = 1.5$. The student uses his "squares" to find how to change a common fraction to





its decimal equivalent. He has learned that every fraction is an indicated division. The fraction $\frac{2}{5}$ means that 2 is to be divided by 5. The diagram shows how the student finds the decimal value of the fraction $\frac{2}{5}$ by the use of objective materials. He discovers that 2 ones cannot be divided as ones, hence the ones are regrouped as tenths. When the 2 ones are regrouped as 20 tenths or 2.0, the example is the same as dividing a decimal by an integer. Diagram A shows 2 ones which are regrouped in B as 20 tenths. In C, the tenths are then divided into 5 equal parts with 4 tenths in each part. Therefore, $\frac{2}{5}$ is equal to 4 tenths, or .4. This fact can be shown with symbols as given below.

$$\tfrac{2}{5} = 5\overline{)2} = 5\overline{)2.0}; \quad \tfrac{2}{5} = .4$$

The student uses objective and visual materials for finding the decimal equivalent of common fractions until he is able to operate with understanding with symbols. He understands the process when he is able to state the steps involved.

The student should use his exploratory materials to find the ratio of a larger whole number to a smaller whole number. First he should express the numbers as a common fraction and then find the ratio between numerator and denominator. He should use small numbers to objectify the process. He should not use manipulative materials to objectify such large numbers as in the example, $20\overline{)75}$.

A	B	C
$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{9}$

The three fractions on the right represent different levels of difficulty in changing a common fraction to a decimal fraction. In A, the numerator of the fraction must be regrouped only once to find the decimal equivalent of the fraction. In B, the 1 of the numerator must be regrouped as 10 tenths and then as 100 hundredths in order to find the exact decimal value of the fraction $\frac{1}{4}$. In C, the exact decimal value of the fraction can never be found. In this case the student must learn how to round off the quotient to the degree of accuracy needed in the particular usage.

The number of places to which a decimal fraction should be expressed depends upon the problem involving the decimal. Many teachers have the student express the ratio of two numbers to two decimal places, rounded off from a third place. If the owner of a car wishes to know the average mileage per gallon of gasoline consumption of his car, the answer expressed to the nearest mile or tenth of a mile would be satisfactory. On the other hand, the average mileage per gallon expressed to this accuracy would be unsatisfactory for expressing the mileage in a scientifically conducted test of gasoline consumption of a motor car. Batting averages and team rankings are illustrations of three-place decimals rounded off from a fourth place. Therefore, the teacher should be sure that the student does not form the conclusion that a decimal whose exact value cannot be found always should be expressed to the nearest hundredth.

Repeating Decimals

The fast learners should be introduced to *repeating decimals* for enrichment of the curriculum. A repeating decimal is one in which the figures in the quotient are the same, as in the quotient $3\overline{)1} = .333\text{---}$, or they repeat in a given sequence, as in the quotient $7\overline{)1} = .142857\text{---}$. The figures in this quotient if extended will repeat in the same sequence. This sequence from one appearance to its next appearance is called the *period* of the repeating decimal. The student should divide to find the period of the figures in the decimal value of the fractions $\frac{1}{7}$ and $\frac{2}{7}$.

Then from a study of the sequence of the digits in the quotient he should be able to write the period for the remaining proper fractions having a denominator of 7. The reader may find it challenging to write the decimal value of the fraction $\frac{3}{7}$, using the period given before for the fraction $\frac{1}{7}$ and the period .285714 for the fraction $\frac{2}{7}$. The superior student should be able to discover the period of a repeating decimal of proper fractions having a denominator of 11. The period of the fraction $\frac{1}{11}$ is .09. From this fact the student should be able to write the period for the other fractions in this family.

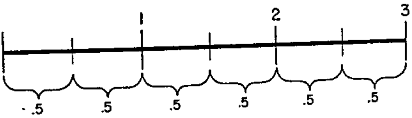
Dividing by a Decimal

Grossnickle[11] gave a test including the four types of examples in division of decimals to 761 students distributed about equally in grades 6–9, inclusive. From the results of the test, the per cent of error and the index of difficulty of each type of example were found to be as follows:

Type of Example	*Per Cent of Total Errors*	*Index of Difficulty*
1. Dividing a decimal by an integer	14.1	0.272
2. Dividing integers with decimal in quotient	17.0	0.781
3. Dividing an integer by a decimal	44.4	1.029
4. Dividing a decimal by a decimal	23.6	0.389

The index of difficulty was found by dividing the number of errors made on each part of the four classifications by the product of the number of students and the number of examples on that part. The data show that dividing an integer by a decimal, as $.3\overline{)9}$, was the most difficult type of example in division of decimals. Just as a student finds it difficult to divide an integer by a common fraction, so he finds it difficult to divide an integer by a decimal fraction. It is very difficult, if not impossible, to make a *visualization* of this operation. A visualization shows the steps of an algorism in terms of the structure of the number

[11] Grossnickle, Foster E. "Type of Errors in Division of Decimals," *Elementary School Journal*. 42:190. Chicago: University of Chicago Press.



system. Any operation which cannot be visualized is difficult, especially for the slow learner. It is easy to make a graphic representation which will show the answer of an example of the type, $.5\overline{)3}$. The diagram shows the answer of this example, but the diagram does not help the student understand the algorism for dividing 3 by .5. A diagrammatic representation of this kind is an *illustration*. An illustration shows the answer but not the steps in the algorism; a visualization shows both the answer and the steps in the algorism. Usually, if a process cannot be visualized, that process should not be part of the basic curriculum for the slow learner. It is possible to visualize dividing a decimal by an integer and an integer by another integer, but it is almost impossible to visualize dividing by a decimal.

A number of different ways may be used to divide by a decimal. In initial learning of the process, the student should make the divisor a whole number by multiplying by a power of 10. It follows that the dividend must be multiplied by the same power of 10. There are two reasons why this procedure should be used in dividing by a decimal. First, multiplying both divisor and dividend by a power of 10 so as to make the divisor an integer applies a mathematical principle which the student understands. Second, if the divisor is an integer, the number of kinds of examples in division of decimals as given on page 212 is reduced from four types to two types. If the divisor is an integer, an example in division of decimals involves dividing either a decimal by an integer or an integer by another integer. The student can objectify both of these types with his kit material and both types can be visualized. Thus, division of decimals is greatly simplified when the divisor is made a whole number by multiplying divisor and dividend by a power of 10.

Every *fraction* is an indicated division, and every division example may be expressed as a common fraction. The example, $.5\overline{)3}$, can be written in the common *fraction form* as $\frac{3}{.5}$. Now it is easy for the student to see that multiplying both terms of the fraction by 10 will make the divisor a whole number. The student should write the steps in the solution as shown. A student will not be able to understand how to make the divisor a whole number until he knows how to multiply by a power of 10 and by what power of 10 a decimal must be multiplied to have the product equal to an integer formed by the digits of the decimal.

$$\frac{10}{10} \times \frac{3}{.5} = \frac{30}{5} = 5\overline{)30}$$

One of the attributes of meaningful learning in arithmetic is growth in dealing with numerical quantities. The student who must continue to rewrite an example in division of decimals so as to make the divisor an integer does not exhibit growth in dealing with this process. He should develop insight into the multiplication process and discover that moving the decimal point towards the right in both divisor and dividend is the same as multiplying both numbers by a power of 10. Then he may use the *shortcut procedure* demonstrated in the example on the right. A caret often is used to show the shift of the decimal-points in the example. Frequently, each shortcut is a mechanical operation, hence it does not necessarily represent growth in the student's ability to deal with quantities. The student should estimate the approximate quotient. In the example, $.15\overline{).135}$, he should see that 15 hundredths is a little larger than 13 hundredths, hence the quotient must be a little less than 1, therefore, .9 is a sensible answer. A student who is unable to determine whether or not an answer to an example in division of decimals is sensible has not reached the level of mastery to be anticipated of most students at the completion of the elementary school. *When a student attains mastery of operations with division of decimals*, he knows that:

$$\begin{array}{r} .9 \\ .15\overline{).135} \end{array}$$

$$\begin{array}{r} .9 \\ .15_{\wedge}\overline{).13_{\wedge}5} \end{array}$$

1. Decimals are divided in the same manner as whole numbers
2. The decimal point identifies ones' place

3. Both divisor and dividend may be multiplied by a power of 10 to make the divisor an integer, which is the same as moving the decimal point towards the right in both numbers, without changing the value of the example
4. He should estimate the answer to see whether or not it is sensible.

In most cases he should estimate the answer and check his estimation by some other procedure. The pattern for approximation can be illustrated in the three examples given below.

A	B	C
$.8\overline{)12}$	$3.2\overline{)27.54}$	$25\overline{)12}$

In A, the student should think, ".8 is a little less than 1, so the quotient would be a little more than 12."

In B, he should think, "27 divided by 3 is 9, so the quotient should be about 9."

In C, he should think, "12 is a little less than half of 25, so the quotient must be a little less than .5."

For enrichment for fast learners, a good check to see that the point in the quotient is placed correctly is to apply the "subtractive principle." According to this plan, the number of decimal places in the quotient is equal to the number of decimal places in the dividend minus the number of decimal places in the divisor. This is the inverse operation of the principle which governs the number of places in the product of two numbers. The number of decimal places in the product of two numbers is equal to the sum of the number of decimal places in these numbers. Thus, there are three decimal places in the product of .15 and 2.5 as shown in A. If the product, .375, with either factor is given, the missing factor is found by dividing the product by the given factor. In B, the given factor is .15 and the missing factor is 2.5. The number of decimal places in the quotient is found by subtracting the

A

$$\begin{array}{r} 2.5 \text{ (1 place)} \\ \times .15 \text{ (2 places)} \\ \hline .375 \text{ (3 places)} \end{array}$$

B

$$.15\overline{).375}^{\,2.5}$$

number of decimal places in the divisor from the number of decimal places in the dividend, or 2 from 3 is 1. Similarly in C, the number of decimal places in the quotient is found by subtracting 1 from 3, which gives 2. A procedure of this kind is desirable for enrichment purposes. As initial instruction, the process is too difficult for most students to learn as a means of finding the position of the point in the quotient.

C $$\begin{array}{r} .15 \\ 2.5\overline{).375} \end{array}$$

Discovering Relationships between Related Processes with Decimals

The same testing procedures that were used with integers and common fractions for discovering the relationship between a process and its inverse should be used with decimal fractions. Samples of the types of examples to be used for this purpose are shown below.

.4	.8	1.2

Addition
(and Subtraction)

.2	.3	.06

Multiplication
(and Division)

The student should be able to make the four examples which can be made with the numbers in each box. He should be able to discover the process to apply to two of the numbers in each box to give an answer equal to the third number. When he discovers how to make one example, he should be able to make the three remaining examples.

d. The Kinds of Problems in Common and Decimal Fractions

The Three Types of Problems in Fractions

The three different usages of common and decimal fractions in problems are:

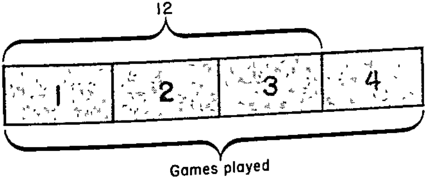
1. Finding a fractional part of a number, as in the problem: Find the cost of $\frac{3}{4}$ pound of salted nuts at $1.60 a pound.

2. Finding the ratio of two numbers, as in the problem: A team won 15 of the 20 games it played. What part of the number of games played did that team win?

3. Finding a number when a fractional part of it is given as in the problem: A team won 12 games which was $\frac{3}{4}$ of the number of games played. How many games did that team play?

These three types of problems correspond to the three usages of per cent discussed in Chapter 7.

The third type of problem, in which the given number represents a fractional part of the unknown number, is usually difficult for most students in the upper grades. A diagram showing the relationship between the given number and the unknown number often helps the student in finding the missing number. The diagram for problem 3 given above is shown below.



The student should see that 3 parts on the diagram represent 12 games, hence 1 of the parts represents 4 games and then 4 parts would represent 4 times as many games, or 16 games. In this case the student uses the method of *unitary analysis* to find the missing number. The solution with symbols would be as follows:

$\frac{3}{4}$ of the number of games = 12
$\frac{1}{4}$ of the number of games = 4 (12 ÷ 3)
$\frac{4}{4}$ of the number of games = 16 (4 × 4)

At a higher level of understanding, the student should be able to find the missing number by dividing the product by the given

fraction. The solution is based on the following mathematical principle which he should understand:

If the product of two numbers and one of the numbers are given, the other number is found by dividing the product by the given number.

In the above problem, the product of two numbers is 12 and one of the numbers is $\frac{3}{4}$. The example may be written as, $12 = \frac{3}{4}$ of ?. The missing number can be found by dividing 12 by $\frac{3}{4}$. Since the divisor is less than 1, the quotient must be larger than 12, the dividend.

The highest level of operation consists in making a formula to express the relationship among the three elements involving fractional usages in problems. If p represents the fractional part, f the fraction, and b the base, the fractional formula becomes $p = bf$. This formula corresponds to the percentage formula which is $p = br$. Buckingham[11] used the term *derivative* to designate the fractional part represented by p in the formula given above. He suggested the letter d to represent the derivative and then the formula would become $d = bf$. It is evident that neither of these formulas should be used until the student has a good background in solving for a missing letter in a formula or an equation. The experience of the writers bears out that most students in the junior high school do not know which numbers in a problem should be substituted for given letters in the percentage formula or in similar formulas. Therefore, the fractional formula should not be used until the student becomes familiar with solving problems involving each type of fractional usage. This formula may be used at the upper level of the junior high school to synthesize the work in dealing with the three fractional usages in verbal problems.

Questions, Problems, and Topics for Discussion

1. What minimum materials should comprise a student's arithmetic kit for dealing with common and decimal fractions?

2. Formulate criteria for determining the kinds of fractions to be taught in addition and subtraction in a course of minimum essentials in arithmetic at the junior high school level.

[11] Buckingham, B. R. *op. cit.*, p. 281.

3. Illustrate several ways of differentiating the curriculum in common fractions.

4. Make samples of all different types of examples possible involving adding two like fractions. The term fractions includes mixed numbers as well as proper fractions.

5. Make samples of all different types of examples possible involving subtraction of two unlike but similar fractions and mixed numbers.

6. What are prime numbers? Write all of the prime numbers from 1 to 100.

7. What is meant by cancellation in multiplication of fractions? What are the basic mathematical principles which govern the operation known as cancellation?

8. Simplify the following complex fractions $\frac{\frac{2}{3}}{4}$; $\frac{4}{\frac{3}{8}}$; $\frac{\frac{7}{8}}{4}$, $\frac{1\frac{1}{2}}{3}$, $\frac{2\frac{1}{2}}{100}$; $\frac{\frac{1}{2}}{\frac{3}{4}}$.

9. State at least five different ways to find the answer to an example in which a whole number is divided by a proper fraction.

10. What is the mathematical reason for inverting a fractional divisor and then multiplying?

11. Show how to divide two unlike fractions by expressing them with a common denominator. Appraise this method of dividing fractions

12. Show how the curriculum in division of fractions can be differentiated for slow learners.

2	$\frac{1}{5}$	6

13. Write four examples using the numbers given in the box.

14. What is the value of a 2 written five places to the left of ones' place? five places to the right of ones' place?

15. Write with figures the number represented as 1200 millions Write the same number expressed as billions. Write the same number in scientific notation.

16. Make a list of at least six different uses of decimals in situations having social significance.

17. Give reasons why the curriculum of a modern school should not include work with ragged decimals.

18. Give the products of the following· a. Tenths and hundredths; b. Tenths and hundreds; c. Hundreds and hundredths; d. Tens and thousandths.

19. Write illustrations of each kind of example in division of decimals.

20. Show the difference between a diagrammatic representation which visualizes a process and one which illustrates it.

21. Many textbooks have given the following directions for changing a common fraction, such as $\frac{1}{4}$, to a decimal fraction: "Annex two zeros to the numerator 1 and insert a decimal point making the number 1.00. Then divide by the denominator, or 4." Appraise this rule from the standpoint of initial learning in finding how to express a common fraction as an equivalent decimal fraction.

22. Enumerate approximately four different ways to divide by a decimal. Appraise each method and indicate the method you consider best.

23. Write four examples using the numbers given in the box.

.8	.4	2

24. What are the three uses of fractions? Give problems to illustrate these three uses.

25. Write the following in numbers:

a. 7.5 millions c. 9.65 millions

b. 3.8 billions d. 7.28 billions

26. Write as indicated:

a. 9,650,000,000 as billions; as millions.

b. 34,800,000 as millions; as thousands.

Suggested Readings

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Chapter 7

Per Cent and Its Applications

THIS chapter deals with the following topics:

a. Teaching the meaning of per cent
b. Finding a per cent of a number
c. Finding what per cent one number is of another number
d. Finding a number when a per cent of it is given
e. Social applications of per cent
f. Compound interest.

a. Teaching the Meaning of Per Cent

Introduction of Per Cent in Grade 7

Most modern textbooks in arithmetic begin the systematic treatment of per cent in the seventh grade. In the '40's the topic of per cent and some of its uses were presented in the sixth grade. There is no inherent difficulty in the topic which would demand that its presentation should be deferred from the sixth grade to the seventh grade. However, per cent has been deferred to the seventh grade in order to have greater assurance of student readiness for the topic than is possible when the subject is presented at a lower grade level. It is not possible to have the treatment of per cent meaningful unless the student has a good background in both common and decimal fractions. The student will probably have a better background in both types of fractions if the time previously devoted to teaching per cent in the sixth

grade is spent in enriching his knowledge of common and decimal fractions.

There is no new mathematical principle to be learned in per cent. However, the language of per cent is new. Thus 25 per cent may be expressed as a decimal fraction, .25, as a common fraction, $\frac{1}{4}$, or as 25%. The number of errors made by students in dealing with per cent would seem to indicate that the topic either is difficult or it is not understood.

A point of minor significance pertains to the writing of the term expressed by the symbol %. This term may be expressed as one word as *percent* or as *per cent*. Both forms are correct and are used, but the term *per cent* is more conventional. The Government Printing Office expresses the term as one word. On the other hand, the terms *per cent* and *percentage* are not synonomous. Students frequently fail to understand the difference between the two terms. A percentage is a quantity which results from finding a per cent of a number. If it is necessary to find 5% of $300, the resulting amount, or $15, is a percentage. The rate is a per cent as indicated by the per cent symbol. A per cent symbol or sign is never used to identify a percentage.

The Meaning of Per Cent

The teacher should introduce the work in per cent by having the students express the ratio of the number of games won to the number of games played in the current year by a local baseball team or some other popular sport. A seventh-grade teacher capitalized effectively on a sporting event to introduce per cent. The lesson was taught the day following the close of a World Series between the Yankees and the Dodgers in which the Yankees won 4 games of the 7 games played. The teacher asked the students to write on the chalkboard the number of games each team won and the number each played. The written statements were as follows:

Yankees won 4 games out of 7.
Dodgers won 3 games out of 7.

Then the teacher asked if anyone could write the same facts in shorter form. An alert member of the class recognized that the fact written in each statement could be expressed as a fraction. The fraction, $\frac{4}{7}$, expressed the ratio of the number of games won to the number of games played. The ratio was read to mean "4 out of 7" and similarly, the fraction, $\frac{3}{7}$, was read correctly to mean "3 out of 7."

If a team wins 7 out of 10 games played, a teacher should have the class write this fact as a common fraction, $\frac{7}{10}$, and interpret it as, "7 out of 10." Then the teacher should have the class find the number of games a team would win at that rate if it played 20 games, 30 games, 40 games, 80 games, and 100 games, respectively. When the students find the number of games each team would win out of 100 games played, the students should be told that the answers found can be expressed in several ways, one of which is as a per cent. Thus, if a team wins 70 games out of 100 games played, we may express the number of games won as 70 *per cent* (%) of the games played.

Using the Hundred Board

A *hundred board* (see page 246) is an effective teaching aid for showing the meaning of per cent. The hundred board consists of a board enclosed by a square frame. The board will hold 10 rows of 10 disks or chips. The chips are movable, so that a student may pick up the correct number of them to represent any given per cent of the whole. Since there are 100 chips on the board, 1 chip will represent 1 per cent of the total number. Thus, one chip out of the group of 100 chips may be expressed as follows:

As a common fraction, or $\frac{1}{100}$ of the total
As a decimal fraction, or .01 of the total
As a per cent, or 1% of the total

Each of these numerical relationships is to be interpreted to mean, "1 out of 100."

The student will discover that if 1 chip out of the 100 chips represents 1%, then 2 chips would represent 2%, and 10 chips

would represent 10%. In the latter case, the student should discover that 10 chips represent $\frac{10}{100}$ of the total, or $\frac{1}{10}$ of it. Therefore, 10 chips out of 100 chips may be expressed as $\frac{10}{100}$, $\frac{1}{10}$, .1, .10, or 10%. Similarly, the per cent value of other common fractions, such as halves, fourths, and tenths, may be identified from the hundred board.

In the first stage of the development of the concept of per cent, the student should use disks on the hundred board to identify or to reproduce per cents. Next, he should identify per cents by use of visual materials. The teacher should have a ruled chalkboard and mark off 100 squares and then represent different per cents on the board. The students should be able to read each per cent indicated. When the student reads a per cent from the drawing, he identifies the number represented.

The next step in the sequential development of the process should consist in having the student reproduce a given per cent. In this case, the teacher should have the students represent a few per cents on the board. Each student should have cross-ruled paper so as to form 100 squares. He should then reproduce different per cents. The amounts to be represented should be expressed in various ways to show the ratio between two quantities. For example, the student should be able to represent as per cents such ratios as 3 out of 100, $\frac{17}{100}$, .19, and $\frac{1}{4}$. In this way he learns that per cent is another way of writing the fractional relationship between two quantities.

Three Different Usages of Per Cent

There are three different usages of per cents in problems which we will discuss in the following pages. These usages are frequently known as the Three Cases of Per Cent. There is no particular advantage for designating these usages as cases. They may be illustrated as follows:

1. Finding a per cent of a number, as 3% of 40 = ?
2. Finding what per cent one number is of another number, as 3 = ?% of 5
3. Finding a number when a per cent of that number is given, as 3 = 5% of ?

The student should understand that he met the same three types of problems in dealing with common and decimal fractions. Representatives of the three usages in common fractions are:

1. Finding a fractional part of a number, as $\frac{3}{4}$ of $20 = ?$
2. Finding what fractional part one number is of another number, as $20 = ?$ part of 30
3. Finding a number when a fractional part of it is given, as $4 = \frac{2}{3}$ of ?

Today very few textbooks in arithmetic introduce the topic of per cent before the seventh grade. The third usage should not be introduced until the eighth grade. There are two reasons for this. First, the topic has limited social applications. Second, the topic is difficult for the student to understand because of his limited experience with per cent. By delaying the topic to the eighth grade, the student has a year in which to enrich his background with the other two applications of per cents Introducing the third usage in the eighth grade should reduce the learning difficulty of the topic for the student.

Use of Per Cents in Newspapers

The teacher should have the students make a collection of the uses of per cents found in the local daily newspaper. These applications of per cent should be classified under the threefold usage of per cent. Ordinarily a study of advertisements of merchandise will provide illustrations of all three uses of per cent. If there is a sales tax in a community, the price of an article may be listed as a given amount plus tax. Thus, a sweater is advertised for $15.50 plus a 3% sales tax. Advertisements of jewelry often give the cost of the article without the federal tax. In all of these cases, the application of per cent involves finding a per cent of a number.

The second usage of per cent involves finding what per cent one number is of another number. This usage may be found in the sports section of a newspaper. Batting averages, team rankings, and pitching records are expressed in "percentages," which are in reality three-place decimals corrected from a fourth place.

In case of advertisements of special sales, the former price and the reduced price of an article frequently are given. The reader of the advertisement may be curious to find the per cent of reduction in the cost of the article. The following table of prices appeared in a newspaper advertisement of imported neckwear:

Formerly	*Now*
\$6.50	\$5.00
5.00	4.00
3.50	3.00
2.50	2.00

The prospective customer should be interested to know in which case the per cent of reduction is greatest.

The third usage of per cent generally can be found in different parts of a daily newspaper. A diamond ring may be advertised to sell for \$440 which would include a federal tax of 10%. The buyer would be interested to know the cost of the ring without tax.

Making problems involving the uses of per cents as found in newspapers constitutes a good class exercise. An advertisement may give the cost of a gold fountain pen as \$15 plus a federal tax of 10% and a state tax of 2%. The student should use these data to make a problem.

An assignment for the superior student should be to find illustrations of the uses of per cent from any part of a newspaper except those in the financial section and in advertisements. An item stated that the increase in cost of operation of the schools in a city for the next year would be \$250,000. This increase represented an increase of 5 per cent in the school costs. From these facts it would be possible to make a problem involving the third usage of per cent.

b. Finding a Per Cent of a Number

Developing the Meaning of the Operation

It is common practice among teachers to introduce a process to a class at the level of operation which characterizes the adult

usage of the process. Frequently, the student is unable to understand the procedure and is obliged to learn by rote. The teacher should provide learning situations that will enable the student to discover the procedure to use in a given operation. The method for finding a per cent of a number typifies the point at issue. The teacher may have the student express the given per cent as hundredths and use this decimal as a multiplier. This process may be meaningful to some of the students but to many of them it is a mechanical rule used for finding an answer to a given problem or example.

The teacher should have the student find a given per cent of 100, such as 6% of 100. By using the hundred board, the student is able to find the percentage. Similarly, he finds other per cents of 100, or the *base*, until he discovers that the percentage of this given number is the same as the number of the per cent. In finding a percentage, the base is the number to be multiplied when the given per cent is expressed as hundredths. The teacher should change the base from 100 to some multiple of 100, as 200, and then have the students use objective material to find the answer to an example, such as 3% of 200, by one of two ways. Either there should be two hundred boards, or there should be two chips on each space on one board. According to the first plan, the student would find 3 per cent on each board, making a total of 6 chips. By the second plan, he would find that 3 per cent means 3 spaces on the board having two chips each, thus making a total of 6 chips. The teacher should have the class find the percentage by expressing the per cent as a common fraction and then as a decimal fraction. To find 3% of 200, the student should work the example as follows:

I. $\frac{3}{100}$ of $200 = \frac{3}{100} \times 200 = \frac{600}{100} = 6$

II. $.03 \times 200 = \begin{array}{r} 200 \\ \times .03 \\ \hline 6.00 \end{array}$

From the illustrations the student should discover that either common or decimal fractions may be used in finding the answer. Since per cent means hundredths or a two-place decimal, it is

easier and quicker in many examples to express the given per cent as hundredths and then to multiply the given base by this number.

The teacher may wish to use exploratory materials for finding a per cent of a number less than 100, as 50. This number can be represented on a hundred board by leaving every other space vacant. If the example involves finding 6% of 50, the student has learned that 6% of 100 means 6 spaces on the board. He counts the first 6 spaces and finds that there are 3 chips on these spaces. Then he uses both common and decimal fractions to find the percentage. Enriched experiences of this kind should enable the student to understand the meaning of finding a per cent of a number and the algorism of the process.

When a student encounters an example of the type, 12% of 45, there should be no attempt to objectify the solution. If a student is ready to solve an example of this degree of difficulty, he should be able to deal intelligently with symbols. If he does not understand the process, he should be given examples which are easy to objectify.

Using Per Cents Less than 1 Per Cent

The number of social applications for finding a per cent less than 1 per cent of a number is limited. It may be necessary to find a fractional part of one per cent in computing interest or in finding a commission on a transaction. Since the number of social applications is limited, the student should learn to use a procedure which will enable him to check on the reasonableness of his answer.

Either of two methods may be used to find a fractional part of one per cent of a number: (1) find 1 per cent of the number and then multiply that percentage by the given fraction or (2) express the fractional per cent as a decimal and then use the same method as that which is used in finding any given per cent of a number. The writers recommend the first of these two procedures because the use of this method enables the student to check on the reasonableness of the answer more easily than by the use of the second method.

A teacher gave the following problem to a group of 67 sophomores in college:

A cargo of 150,000 bushels of wheat, valued at \$2 a bushel, was insured at the rate of $\frac{1}{8}\%$ of its value. How much was the insurance premium?

There were only 36 correct answers to the problem. Two of the students found the premium to be \$3,750,000 These students changed $\frac{1}{8}\%$ to 125. Apparently they omitted the decimal point. Then they multiplied \$300,000, the value of the wheat, by 125. Most of the incorrect answers resulted from the faulty placement of the decimal point in the answer. The students did not know if the decimal value of $\frac{1}{8}\%$ should be expressed as .0125 or as .00125. The answer found by multiplying \$300,000 by either of these two ratios would seem sensible to most students On the other hand, there was not one incorrect answer given by a student who solved the problem by first finding the premium at the rate of 1 per cent and then multiplying that percentage by $\frac{1}{8}$. The student should be able to find 1 per cent of a number by dividing the number by 100. He should know that this is the same as moving or placing the decimal point two places to the left in the number. He can do this part mentally. Then he finds the fractional part of the 1% he found mentally.

The teacher should not use objective materials to show how to find a fractional part of 1 per cent of a number, but the hundred board can be used to show the meaning of a fractional part of 1 per cent. If 1 chip on a hundred board represents 1%, then half a chip would represent $\frac{1}{2}$ of 1%, or $\frac{1}{2}\%$. It follows then that 1 chip out of a group of 200 chips is the same as $\frac{1}{2}$ per cent. Exploratory materials are not needed to objectify the uses of these small per cents. Problems involving these per cents are beyond the scope of minimum essentials for slow learners. If a student is unable to deal with symbolic materials, he should not be required to master subject matter which has such a limited social application as finding a fractional part of one per cent of a number. Show this student how to find a fractional per cent of a number, but do not require mastery of the process by this slow student.

A Social Application of Finding a Fractional Per Cent of a Number

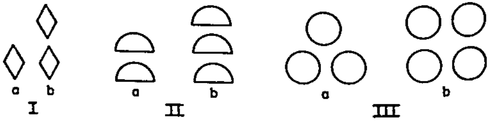
An example of one of the social applications of a fractional part of 1 per cent consists in finding the cost of excess baggage which a passenger transports on an airliner. Frequently, a passenger is allowed to transport free 40 pounds of baggage. For each pound of baggage in excess of 40 pounds, the charge is $\frac{1}{2}\%$ of the fare. If a passenger has 50 pounds of baggage and the fare is \$48, the cost of transporting the baggage is 10 times $\frac{1}{2}\%$ of \$48. Since 1% of \$48 is 48¢, $\frac{1}{2}\%$ of \$48 is 24¢; then 10 × 24¢, or \$2.40, is the extra cost for the baggage.

The method shown is the procedure used for instructional purposes for the class. In order to provide enrichment for the superior students, the teacher should encourage the fast learners to solve the problem in one or more ways. Some of the students may change $\frac{1}{2}\%$ to its decimal equivalent, .005, and multiply \$48 by this ratio. The product, 24¢, would then be multiplied by 10. One or more of the students may multiply .005 by 10, giving a product of .05. Then .05 × \$48 would be the cost for transporting the baggage. The student giving this solution should understand the principle that the order in which factors are multiplied does not affect the product. Finding different solutions to problems challenges superior students.

c. Finding What Per Cent One Number Is of Another Number

The Meaning of Ratio

The *ratio* of two quantities may be expressed as a per cent. In the diagram, the ratios of a to b are $\frac{1}{2}$, $\frac{2}{3}$, and $\frac{3}{4}$. Each ratio may be expressed as hundredths and finally as a per cent.



The procedure to follow to find what per cent one number is of another number may be shown by solving the following problem: A team played 20 games and won 13 of them. What per cent of the games played did it win? If the student has difficulty in solving the problem, the teacher should have him use the hundred board to find the answer. A chip may be used to represent a game. If 20 games were played, 20 chips must be distributed equally on the hundred board. One chip then represents 5 spaces. Then the 13 chips to represent the number of games won would represent 65 spaces, or 65 per cent of the number of spaces on the board; hence, the team won 65 per cent of its games. The same answer can be found by expressing as a two-place decimal the ratio of the number of games won to the number of games played. This ratio is $\frac{13}{20}$ and this fraction can be expressed as hundredths as shown on the right. Then the decimal, .65, may be written as 65 per cent.

$$\frac{13}{20} = 20\overline{)13.00}\quad .65$$
$$\begin{array}{r} 12\,0 \\ \hline 1\,00 \\ 1\,00 \\ \hline \end{array}$$

The difficult part for the student is to express the ratio of the two given quantities. Often he does not know which of the two numbers should be the numerator of the fraction or which number should be the denominator. Teachers sometimes direct their students to look for "cues," such as the number following "of." This number will be the denominator of the fraction forming the ratio and the other number must be the numerator. A procedure of this kind is distinctly mechanistic and lacks meaning to the student.

The student should be taught to ask himself the following:

> What number is to be compared?
> With what number is it to be compared?
> Could the answer be more than 100 per cent?
> Must the answer be less than 100 per cent?

The number to be compared is the numerator of the fraction expressing the ratio of the two numbers. The number with which the comparison is made is the denominator of the fraction. In the above problem, the student should see that the number of games won, or 13, is to be compared with the number of games

played, or 20. The per cent of games won must be less than 100%, therefore, the student has a check to know that the ratio of the two numbers must be $\frac{13}{20}$ and not $\frac{20}{13}$. Many students are unable to express the correct ratio between two quantities in solving problems which involve finding what per cent one number is of another number.[1] This failure results from a lack of meaningful experience in dealing with quantities. The setting of the problem is unfamiliar, hence the student is unable to determine whether the answer could be more than 100 per cent or whether it would have to be less than 100 per cent. The best way to help this type of student is to enrich his background about a given problem. Either he must have an enriched background which will enable him to apply mature judgment to a problem or he must depend upon mechanistic cues as aids in its solution.

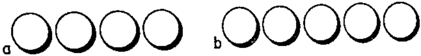
Per Cents Greater than 100 Per Cent

The student learns that 100 per cent of a thing is all of it. It is difficult for him to understand how there can be more than 100 per cent of a number. In a recent presidential campaign, one of the candidates stated, "The steel mills of this country are operating at 110 per cent of capacity." In finding what per cent one number is of another number, the teacher should not have the students generalize that the smaller of two numbers is always compared with the larger number.

A per cent greater than 100 per cent will result only when there is a comparison of two amounts or a change in a quantity. The use of disks taken from a hundred board or chips is effective for showing a per cent greater than 100 per cent. Each student should form two stacks of disks and find the ratio of the two quantities in both orders. Then he should express each ratio as a per cent.

[1] See Edwards, Arthur "A Study of Errors in Percentage," *Report of the Society's Committee on Arithmetic*, pp. 635–636. The Twenty-ninth Yearbook of the National Society for the Study of Education Chicago: University of Chicago Press, 1930.

Guiler, Walter S. "Difficulties Encountered in Percentage by College Freshmen," *Journal of Educational Research*. 40.81–95.

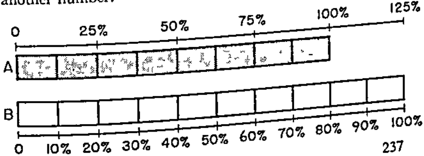


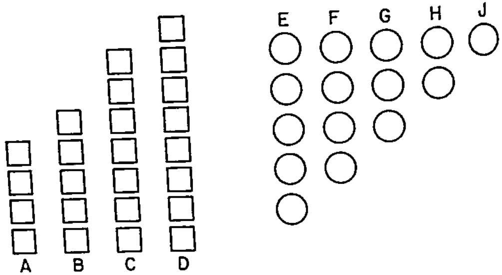
In the diagram above, the ratio of the number in a to the number in b is $\frac{4}{5}$ or 80%, and the ratio of the number in b to the number in a is $\frac{5}{4}$ or 125%. The student should vary the number of disks in each stack so as to enable him to make the following generalizations:

1. When a smaller number is compared with a larger number, the smaller number is less than 100 per cent of the larger number.
2. When a larger number is compared with a smaller number, the larger number is greater than 100 per cent of the smaller number.
3. When two equal numbers are compared, one number is 100 per cent of the other number.

The next step in the sequential development should consist in interpreting per cents greater than 100 per cent from visual materials, such as graphs. In the given graph, bar A is only 80% as long as bar B, but bar B is 125% as long as bar A. The teacher should have the students see that 80% is 20% less than 100% and that 125% is 25% more than 100%. The student with insight should discover why the differences from 100 per cent are not the same. In bar A, the whole is divided into 8 equal parts, but in bar B, the whole is divided into 10 equal parts. Since bars A and B are both wholes, 2 parts on bar A would represent a greater number than 2 parts on bar B.

Finally, the student should work with symbols. He should compare quantities and find the per cent one number is of another number.





Finding the Per Cent of Increase or Decrease

The per cent of increase in the cost of a commodity can be 100%, less or greater than 100%, but the per cent of decrease based on the cost can never be 100 per cent, if the article is sold. Unless exploratory materials are used, many students are unable to understand why the change cannot be 100 per cent in both increase and decrease in cost. The use of objective materials, such as squares or disks on a flannel board, should enable the student to discover why the per cent of increase in cost may be unlimited, but the per cent of decrease in cost must be less than 100 per cent, as long as the article is not given away.

Let the four squares in A represent the cost of a commodity. The student should know that each square represents 25%. If he does not see why this is true, he should space the squares equally on a hundred board. Then he will discover that each square represents 25 spaces on the board, or 25 per cent of it. Columns B, C, and D show the increased costs at different periods of time. The per cent of increase from the original amount in A as shown in B is 25%, in C it is 75%, and in D it is 100%. The student should see that another increase in cost would produce a total increase greater than 100 per cent.

The five circles in column E represent the original cost of a commodity, hence each circle represents 20%. Columns F, G, H, and J represent decreased costs at different periods of time.

The per cent of decrease from the original cost in E as shown in F is 20%, in G it is 40%, and H it is 60%, and in J it is 80%. To have a decrease of 100%, all of the circles would have to be removed. In that case there would be no charge for the commodity. An objective demonstration of this kind should enable the student to discover why the selling price of an article can never be reduced 100 per cent of its former price. The sample given refers to the cost of a commodity. The per cent of change may apply to any measurable quantity, such as population, weight, or volume.

The teacher should have the student discover the difference between the per cent of increase in cost and the per cent the increased cost is of the original cost. Referring to columns A and D in the diagram on page 238, the per cent of increase in cost from A to D is 100 per cent, but the cost in D is 200 per cent of the cost in A. When the population of a city doubles, the per cent of increase in population is only 100 per cent; however, the population of the city is 200 per cent of its former population.

d. Finding a Number When a Per Cent of It Is Given

Why This Operation Is Difficult

Some steps in arithmetic are difficult to learn regardless of the type of instruction given. As was shown in Chapter 3, finding a number when a per cent of it is given represents an inherent difficulty for most students. This applies not only to students in the elementary grades, but also to students in high school and college. Guiler made a study of the results from a survey test given to 936 students in the ninth grade. The results showed that nine-tenths of these students had difficulty in that part of the test in which a number is found when a per cent of it is given.[3]

The lack of experience which the student has with social situations involving problems dealing with this usage of per cent is largely accountable for the difficulty of these problems. One of the criticisms of many problems in this category is their lack of

[3] See Guiler, Walter S. "Difficulties in Percentage Encountered by Ninth-Grade Pupils," *Elementary School Journal*. 46:563. Chicago: University of Chicago Press.

reality. The person making the problem often knows the answer before he formulated the problem as shown by the following: An advertisement states that a rail ticket is offered for $15 at 60% of the regular price. What is the regular price? The company knows the regular price, but you as the consumer do not know it. If you have intellectual curiosity, you would like to know how much the regular fare would be. Often an advertisement gives the selling price and the discount as shown. The prospective customer should be interested in the former price which was $15. The solution of this problem represents the third usage of per cent. The student cannot have a full mathematical understanding of the topic of per cent if one of its uses is not developed. It is not a question whether or not the topic should be taught. The vital problems pertain to the manner in which the topic should be taught and when it should be introduced in the curriculum.

> Dresses now
> $12
> at 20% off

Three Kinds of Problems Representing the Third Usage of Per Cent

Three kinds of problems involve the third usage of per cent.

1. Finding a number when the number given represents the per cent given. Illustration: A subscription to a magazine at 80% of the regular price is $4. What is the regular price?

$$80\% \text{ of regular price} = \$4, \text{ or } .8 \times ? = \$4.$$

2. Finding a number when the per cent given must be subtracted from 100% to represent the number given. Illustration: The cost of a radio at 20% off was $24. What was the regular price of the radio?

$$100\% - 20\% = 80\%. \quad 80\% \text{ of regular price} = \$24,$$
$$\text{or } .8 \times ? = \$24.$$

3. Finding a number when the given per cent must be added to 100% to represent the number given. Illustration: The cost of a watch including a 10% tax was $55. What was the cost of the watch without the tax?

$$100\% + 10\% = 110\%. \quad 110\% \text{ of cost of watch} = \$55,$$
$$\text{or } 1.1 \times ? = \$55.$$

It is easily seen that the second and third problems in this group include an extra step which greatly complicates the difficulty of these problems. Many students erroneously solve the third problem by finding 10% of \$55, or \$5.50 and then subtracting this amount from \$55 to find the cost of the watch to be \$49.50 instead of the actual cost of \$50.

The purpose of spacing a topic, such as per cent, through several grades is to provide optimum conditions for ease of learning the topic or process and to keep the instructional load of any one grade from becoming unbalanced by difficult steps in that process. In light of these considerations, it is defensible to defer the second and third types of problems in the third usage of per cent to the ninth grade. At this grade level, the average student should be able to understand formulas and equations and how to work with them. Then he should be able to use the *percentage formula*, $p = br$, and he should use this formula in solving all problems in dealing with per cents. In this formula, p = percentage, b = base, and r = rate of per cent. In the third usage of per cent, the number representing the base is missing. The formula for finding the base is $b = \frac{p}{r}$. When the student has the background to understand the formula, he is able to form a unified concept of the topic of per cent which he frequently does not get when he studies the three usages independently of each other. In light of these factors, the new work in per cent should be spaced through Grades 7, 8, and 9. A defensible program should have the average student in Grade 7 deal with the first and second uses of per cent. In Grade 8 he should deal with per cents less than 1 per cent and with finding a number when the given per cent represents the given number. Finally, in Grade 9 he should deal with all of the usages of per cent. At this grade level he should use the percentage formula, $p = br$, to solve for any of the three variables in the formula.

How to Find the Missing Number

Two methods may be used to find a number when a per cent of it is given. The one method may be designated the *method of*

unitary analysis and the other may be designated the *product-factor method.*

According to the method of unitary analysis, the student should find 1% of the number and then multiply this amount by 100 to find the missing number. The method may be illustrated in the solution of the following problem: If a radio sold for \$16 at 80% of the regular price, what was the regular price?

If 80% of the regular price = \$16
1% of the regular price = \$.20 (\$16 ÷ 80 = \$.20)
100% of the regular price = \$20 (100 × \$.20 = \$20)

The problem can be solved in a similar manner by changing 80% to the equivalent fraction $\frac{4}{5}$. Then the solution would be as shown:

If $\frac{4}{5}$ of the regular price = \$16
$\frac{1}{5}$ of the regular price = \$ 4 (\$16 ÷ 4 = \$4)
$\frac{5}{5}$ of the regular price = \$20 (5 × \$4 = \$20)

The method of unitary analysis is easy for the student to understand, but it is a long procedure. Frequently, students shorten the solution so that the work takes the undesirable form shown on the right. Work of this kind must not be tolerated. A per cent represents a part of a quantity or the ratio of two quantities. It is not possible for 80% to equal \$16. It is 80 per cent of the regular price which is equal to \$16.

80% = \$16.
1% = 20¢
100% = \$20

The product-factor method is based on the principle that when the product of two numbers and one of the numbers are given, the other number can be found by dividing the product of the two numbers by the given number. Thus, in the expression, 3 × ? = 12, the product, 12, and one of the factors, 3, are given. The missing factor can be found by dividing 12 by 3. In the same way the problem given above may be solved as follows:

80% = .80 = .8
If .8 × the regular price = \$16
Then, \$16 ÷ .8 = the regular price, or \$20

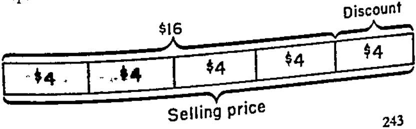
The advantage of this method is twofold. First, the solution is easy and direct. Second, the method is an application of a

basic mathematical principle which the student should understand. (See principle No. 5 on page 144.) The teacher can always refer to an example of the type, $3 \times ? = 12$, to have the student understand the procedure to follow. The student should recognize that he would find the missing number by dividing the product by the given factor. In the problem under consideration, the product of two factors is \$16. The given factor is .8; hence the other factor can be found by dividing 16 by .8. The answer of \$20 is sensible because 16 is divided by a number less than 1; therefore, the quotient must be greater than 16 When the given factor is greater than 1, as in problem 3 on page 240, the quotient must be less than the number divided. In problem 3, the product of two numbers is 55 and one of the factors is 1.1, therefore, the quotient of 55 divided by 1.1 would be the other factor, or 50. The answer, \$50, is sensible because the divisor is slightly greater than 1. When the divisor is slightly greater than 1, the quotient must be a little less than the number divided.

Use of Visual Aids in Solving Per Cent Problems of This Kind

A diagram should be an effective visual aid in the solution of problems dealing with the third usage of per cent. The diagram shown is an effective aid in the solution of the problem given on page 242.

The ratio of the reduced price to the regular price is $\frac{4}{5}$, hence a diagram to show this relationship should be divided into 5 equal parts. Then four of these parts would represent the reduced price, or \$16. Each part of the diagram represents \$4 and the whole diagram would represent \$20. The symbolic solution is on a higher level of abstraction than the solution employing a visual representation.



In a problem of this type, if the given per cent is expressed as a common fraction, the student usually has no difficulty of knowing into how many equal parts he should divide a diagram to represent the relationship between two quantities. If a student is unable to give a symbolic solution to a problem in this category, he should make a visual representation of the data. It is doubtful whether objective materials should be used as aids in the solution of problems in this group. Problems representative of this application of per cent are beyond the level of the minimum program for all students. The use of exploratory materials should be confined to teaching concepts and procedures which are part of the minimum program for all students.

Showing Relationships between Fractions and Per Cents

The teacher should provide frequent opportunities for the student to discover relationships between fractions and per cents. The following problems show how a problem of each usage in fractions may be expressed with per cent or in the reverse form. A decimal fraction may be substituted for a common fraction.

I. Kinds of Problems with Fractions	II. Kinds of Problems with Per Cents
1. Finding a fractional part of a number	1. Finding a per cent of a number
Dick had \$4.50 and spent $\frac{3}{5}$ of it for a baseball. How much did he spend for the ball?	Dick had \$4.50 and spent 60% of it for a baseball. How much did he spend for the ball?
$\frac{3}{5}$ of \$4.50 =	60% of \$4.50 =
2. Finding what part one number is of another number	2. Finding what per cent one number is of another number
A team played 20 games and won 14 of them. What part of the games played did that team win?	A team played 20 games and won 14 of them. What per cent of the games played did that team win?
$14 \div 20 =$	14 = ______% of 20

3. Finding a number when a fractional part of it is given

A jacket was sold for \$24 at $\frac{4}{5}$ of the regular price What was the regular price?

$\frac{4}{5}$ of ______ = \$24

3. Finding a number when a per cent of it is given

A jacket was sold for \$24 at 80% of the regular price. What was the regular price?

80% of ______ = \$24

A student who has had all three usages of per cents should be given problems which are representative of these three types. He should be able to classify each problem and then solve it by using both common fractions and per cents. An exercise of this type should help him understand how fractions and per cents are interrelated.

When the student understands how to use the percentage formula, $p = br$, he should be given problems to solve involving all three types of per cent by the use of the formula. In each problem, he should identify the values for two of the letters in the formula. Substituting these values in the formula, an equation would be formed. Then he should solve the equation for the missing value.

In problem 3 above, the values given are p = \$24 and r = 80% or .8. Substituting these values in the formula $p = br$, the equation and its solution become

$$24 = .8b$$

$$\text{or} \quad .8b = 24$$

Dividing both terms by .8: $b = 30$

The regular price of the jacket was \$30.

Diagnostic Test on Per Cent Computations

See pages 246–247 for a comprehensive diagnostic test which contains illustrations of socially usable examples in per cent. Parts of the test can be used to discover learning difficulties after the presentation of each phase has been taught. It also provides an excellent end test for Grade 8 or a pretest to be used in planning review work for Grade 9.

DIAGNOSTIC TEST IN PER CENT

I. Express the following in decimal form:

1. 30%	2. 200%	3. 125%	4. 3.5%	5. 4%

II. Change the following to per cents:

1. .4	2. .52	3. 1.03	4. .025	5. 3

III. Change to common fractions expressed in lowest terms:

1. 75%	2. $37\frac{1}{2}$%	3. 40%	4. 5%	5. $66\frac{2}{3}$%

IV. Change to per cents:

1. $\frac{1}{4}$	2. $\frac{5}{8}$	3. $\frac{3}{20}$	4. $\frac{6}{25}$	5. $\frac{7}{50}$

V. Find the per cent of the number as indicated:

	a	b	c
1.	4% of 60 =	15% of 80 =	70% of 25 =
2.	3.2% of 95 =	102% of 560 =	300% of 55 =
3.	$\frac{1}{2}$% of 480 =	$1\frac{1}{2}$% of 920 =	$\frac{1}{4}$% of 1200 =

If the results of a diagnostic test show that reteaching is necessary, go back to the basic definitions using a hundred board.

Demonstration School, State Teachers College, Jersey City, New Jersey

VI. Find what per cent one number is of another number, as indicated:

	a	b	c
1.	4 = ? % of 16	15 = ? % of 45	12 = ? % of 40
2.	15 = ? % of 6	12 = ? % of 8	70 = ? % of 35

VII. Find the number when the per cent of it is given, as indicated:

	a	b	c
1.	5% of ? = 80	15% of ? = 75	3% of ? = 60
2.	110% of ? = 55	$60 = 120% of ?	600 = 150% of ?

If a student has only one example incorrect in any group, he should correct the error. If he has two or more examples incorrect in any set of examples, he should review that particular phase of per cent.

e. Social Applications of Per Cent

Everyday Uses of Per Cents

The social applications of per cent studied in the upper grades are of many different kinds. Though ratios sometimes are used, the most common way to compare two numbers is to find what per cent one number is of the other number. This usage is prevalent in sports and in expressing the amount of change which takes place in a quantity under certain conditions. Two of the various other familiar applications of per cent include interest and merchandising. Interest may be considered as *simple interest* as used in commercial banks or as *compound interest* as used in savings banks. Merchandising refers to marketing goods and includes such concepts as *trade discount, mark-up, margin, commission, profit, loss,* and *instalment buying.* In most of these social applications the understanding of the concepts used constitutes the chief element of difficulty and the usage of per cent in the upper grades is either finding a per cent of a number or finding what per cent one number is of another number. The student has to find either the percentage when the base and the rate are given,

or the rate when the percentage and the base are given. With the exception of instalment buying, the student should not find it difficult to identify the given elements.

Simple Interest

If you rent a car, you pay for the service which it renders. Similarly, if you rent or borrow money, you should pay for the service which it renders. The charge for this service is known as *interest*. Interest is understood to mean simple interest to distinguish it from compound interest. An individual may have money to rent or lend, or it may be necessary for him to rent or borrow money. In the one case he should receive interest on the loan and in the latter situation he should pay interest on the money borrowed. A bank is the most familiar institution which pays interest on money deposited and charges interest on money loaned.

Applications of finding interest on loans or principals are familiar usages of finding a per cent of a number. Thus, the interest on \$300 at 2% for 1 year is the same as finding 2% of \$300, or \$6. If the period of time is for 2 years, the interest would be twice as much as for 1 year. Similarly, the interest for a half year would be half the interest for 1 year. In initial learning of the topic, the student should always be taught to find the interest for one year and then multiply that amount by the time expressed in years or in the fractional part of a year. Find the interest on \$500 at 2% for 1 yr. 6 mo. as follows:

1 yr. 6 mo. $= 1\frac{1}{2}$ yr.
2% of \$500 $= .02 \times \$500 = \$10.00$
$1\frac{1}{2} \times \$10 = \frac{3}{2} \times \$10 = \$15$
The interest on \$500 at 2% for $1\frac{1}{2}$ yr. is \$15.

In the above illustration, the student identifies \$500 as the *principal*, 2% as the *rate*, and $1\frac{1}{2}$ years as the *time*. From illustrations of the type given, he should discover that

$$\text{interest} = \text{principal} \times \text{rate} \times \text{time}$$

or $$i = p\,r\,t$$

The student should understand the *interest formula*, $i = prt$, as well as he understands the formula for the area of a rectangle. He should not substitute blindly in the formula, $i = prt$, to find interest. The formula should mean to him that principal is to be multiplied by rate and that percentage is to be multiplied by the number representing the time.

Finding Interest for Short Periods of Time

Our calendar year is not well adapted for computing interest because neither 365 nor 366 is a multiple of either 12 or 30. Banks use a year of 360 days, because 360 is a multiple of 30, 60, and 90.

The most common period for computing interest on loans at banks is 30, 60, or 90 days. However, if a loan from a bank is made on a certain date and it is to be paid 3 months later, interest is charged for the actual number of days in that interval. For example, the interest on a loan made March 1 to mature in 3 months would be computed for a period of 92 days.

It is a common practice to teach students in the upper grades to find interest on a loan by substituting the correct values in the interest formula, $i = prt$. The use of cancellation should simplify the computation, but this procedure may increase the difficulty of the process. To illustrate: Find the interest on a loan of \$800 at 6% for 45 days.

$$i = prt, \text{ the formula}$$

$$i = \overset{\$2}{\overset{\cancel{\$8}}{\cancel{\$800}}} \times \frac{\overset{1}{\cancel{6}}}{\underset{1}{\cancel{100}}} \times \frac{\overset{3}{\cancel{45}}}{\underset{\underset{1}{\cancel{4}}}{\underset{\cancel{60}}{\cancel{360}}}} = \$2 \times 3 = \$6$$

The great number of divisions as shown by the cancellations offers many possibilities for error. By using this method, the student has no check on the reasonableness of the final result. The steps to be followed in solving the example are: (1) find

the interest for 1 year; and (2) multiply the yearly interest by the time represented by the fractional part of one year. The fraction to represent the time should be expressed in lowest terms. We may solve the example given above as follows:

$$45 \text{ da.} = \tfrac{45}{360} \text{ yr.} = \tfrac{1}{8} \text{ yr.}$$

(1) Find the interest on \$800 at 6% for 1 yr.

$$.06 \times \$800 = \$48.00$$

(2) $\frac{1}{8} \times \$48 = \$6$, the interest on \$800 at 6% for 45 da.

The answer given is sensible. The student knows that the yearly interest is \$48, hence the interest for $\frac{1}{8}$ year should be \$6. He uses the interest formula to give the sequence of steps in the solution. When the fractional part of a year is an irreducible fraction, such as $\frac{31}{360}$, sometimes it is easier to complete the solution by the method employing cancellation than by the other procedure. Even though the method used may be difficult in these few cases, the teacher should have the students use a uniform procedure for finding interest on short term loans. When the student thoroughly understands the procedure and is able to approximate a sensible answer, then he should be permitted to use a shortcut procedure, such as substituting in the formula and then reducing the numbers by cancellation. In initial learning, the process which emphasizes meaning and understanding should always be used.

Interest on Personal Loans

Banks make different kinds of loans. A *personal loan* is a loan which is not secured by collateral or by other signatures. The maximum amount of a personal loan varies from one bank to another, but the amount usually is not in excess of \$1000 and seldom is loaned for more than 6 months. A bank issuing a loan of this kind *discounts* it in advance. If Mr. Brown makes a personal loan at a bank for \$1000 at 6% for 6 months (180 days), the bank will pay him \$970 and not \$1000. The interest is collected in advance and it is called *bank discount*. Bank discount is collected

on personal loans and on loans of a specified period of time, such as those having durations of 30, 60, or 90 days.

From the above discussion it is seen that the difficult part of finding simple interest consists in understanding the meaning of the terms used in the applications of interest. The mathematics of finding interest involves finding a per cent of a number. The technical terms used in the social applications of simple interest are not within the experience of most students. The teacher should arrange excursions to banks or plan to have representatives of such institutions visit the class to discuss such topics as loans and under what conditions they are made. It would be advisable to have the representative of a bank distribute samples of the different blanks used. Much of the work dealing with social applications of arithmetic in the upper grades is informational and requires a minimum of computation.

Concepts Used in Merchandising

Arithmetic was introduced into the schools of this country primarily to enable the student to enter into a commercial enterprise. This was especially true of the schools founded by the early settlers of New Amsterdam. Problems arising from merchandising of goods have been in the arithmetic textbooks of the United States ever since the first books appeared in this subject.

Although concepts used in merchandising have had a traditional place in the curriculum in arithmetic of the upper grades, students in these grades have a more meager understanding of these concepts than of any other group of concepts found in the social applications of arithmetic.[3] The reasons for this lack of understanding of the concepts are twofold: first, these concepts are not within the experience of most students and second, these concepts are representative of a specialized field of business. It is natural that problems dealing with the social practices within an enterprise which is unfamiliar to most students would employ many meaningless concepts to these students. If the terms and

[3] Grossnickle, Foster E. "Concepts in Social Arithmetic for the Eighth Grade Level," *Journal of Educational Research*, 30:481.

concepts used in arithmetic are not understood, the work in that subject can never be satisfactory in a program which stresses meaningful arithmetic. The vital problem of the teacher dealing with these concepts is to try to provide experiences which will enable the student to give meaning and understanding to these terms.

Merchandising Concepts Presented in Arithmetics

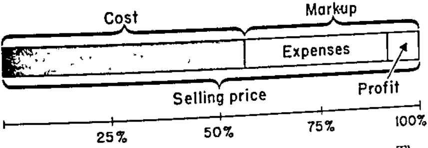
Textbooks in arithmetic differ greatly in the concepts which the student is expected to learn while dealing with the topic of merchandising. Any concept taught should agree with the concept in actual business usage. The following concepts may be considered as a minimum for this topic: *cost*, *mark-up* or *margin*, *profit*, *loss*, *expenses*, *list price*, *discount*, *commission*, and *instalment buying*. The term *retail* is used in business practice to designate selling price. An analysis of current textbooks in arithmetic will show that the term selling price is used almost exclusively in preference to "retail."

It should be evident that the student will have difficulty in learning so many concepts which are largely unfamiliar to him because he seldom encounters them in his daily affairs. Furthermore, it is difficult to use exploratory materials to objectify their meaning. There is no mathematical principle or relationship which the student can discover in the use of these terms. Each term expresses a social application of number which must be defined for the student.

One of the basic formulas for the student to know deals with selling price, cost, and mark-up or margin. This formula may be stated as follows:

$$\text{Selling price} = \text{cost} + \text{mark-up}$$
$$\text{or,} \quad \text{Cost} = \text{selling price} - \text{mark-up}$$
$$\text{or,} \quad \text{Mark-up} = \text{selling price} - \text{cost}$$

Both the teacher and the students should understand that only one statement of relationship is necessary. The second and third statements given are derived from the first statement by subtraction.



The graph shows the relationships among the three terms. The terms mark-up and margin are used interchangeably, but mark-up should be the preferred usage. Since mark-up is the difference between selling price and cost, this difference includes expenses of operation of the business. If the expenses are less than the mark-up, there is a profit; if the expenses are greater than the mark-up, there is a loss.

Teachers should understand that the accepted retail practice is to express mark-up, expenses, and profit as a per cent of the selling price and not as a per cent of the cost. It is much easier to use sales price as a base than to use cost when taking an inventory. Then, too, commissions are paid on sales price and not on the cost. The diagram above shows that it is easier to see the relationship among the various parts when one of the elements constitutes the whole. If there are no expenses in the operation of a business, as in the case of the business of a newsboy who buys and sells papers, the profit may be expressed as either a per cent of the cost or of the selling price. In all business organizations there are expenses. It is mathematically possible to express mark-up, expenses, and profit as a per cent either of the cost or of the selling price, but in order to conform to business practice these per cents should be expressed as part of the selling price.

Discount and Successive Discounts

The term *discount* or *trade discount* implies that the list price is marked down. Discount is another term for *markdown*. Discount is given for stimulating trade, for encouraging a quick turnover of stock, or for many other reasons. Most of the mathematical

applications of discount involve either finding a per cent of a number or finding what per cent one number is of another number. If the per cent of discount is given, the application consists in finding a per cent of a number. If the list price and the sales price are given, the application consists in finding the per cent of discount.

Sometimes two or more discounts are given from the list price. These discounts are usually known as *successive* or *chain discounts.* They apply chiefly to wholesale trade and not to retail trade. The student should be taught to find the net cost in the same manner that he has been taught to find the net cost when only one discount is given. He should find the net cost after deducting one discount and continue the process for each succeeding discount.

Often a student does not understand that the order in which discounts are deducted does not affect the net cost. Thus, discounts of 20% and 5% have the same value as discounts of 5% and 20%. The student should find the net cost of a given article at a certain list price when two discounts are allowed by reversing the order of the discounts. This procedure will prove to him that the order in which the discounts are deducted does not affect the net cost, but he may not understand why this is true. The mathematical reason can be shown by finding the net cost by using the complement of each discount. The method may be illustrated by the following problem:

Find the net cost of a chair listed at \$150 less 20% and 10%.

If 20% of the list price is discount, then 80% of the list price would be cost. Similarly, 90% of the list price would be cost. Then the net cost would be found as follows:

$$.8 \times .9 \times \$150 = \$108$$

$$\text{or} \quad .9 \times .8 \times \$150 = \$108$$

The student has learned that the order of multiplying factors does not affect the product. He should understand that it will make no difference in finding the net cost in which order the discounts are deducted.

On the other hand, he should perform experiments to show that successive discounts of 20% and 5% are not the same as one discount of 25%.

Instalment Buying

The cost of an article purchased on the instalment plan is usually paid in equal instalments at equal intervals, such as monthly or weekly. The merchant extends credit to the purchaser of an article and for this service the merchant charges interest. The rate charged may be much greater than the rate charged by a bank. The interest charge makes the purchase price of an article bought on the instalment plan greater than the cash price of the article.

There are advantages and disadvantages of instalment buying. The teacher should have the class discuss some of the pros and cons of this form of purchasing Contrary to popular opinion, only a small percentage of the articles purchased on the instalment plan have to be reclaimed. Approximately one per cent of these purchases have to be reclaimed; therefore, the loss incurred in this form of credit is insignificant in affecting the rate charged for instalment credit.

The established rate of interest charged on a loan paid off in equal instalments varies with the method used in computing the rate. Actually, the debt incurred by an instalment purchaser is amortized over a fixed period of time and each payment is partly principal and partly interest. In determining the true rate of interest, due consideration must be given to the factor of the component parts of each payment. The method of determining the true rate is beyond the scope of a course in mathematics at either the elementary or secondary level.

The experience of the writers has proved to them that most students at the junior high school level should not be required to compute the rate of interest charged on instalment loans. The student should find the interest charge on a loan. This charge is the difference between the cash price and the instalment price.

Finding the Approximate Rate on Instalment Loans

One method of finding the approximate rate of interest on an instalment loan consists in comparing the interest charged for the loan with "the average amount of the loan." The average amount

of the loan may be considered the average of the unpaid balance (the instalment price less the down payment) and the last payment.[4] If the purchaser of an article bought on the instalment plan owed \$80 which was to be paid off in eight monthly instalments of \$10 each, a \$45-loan $\left(\frac{\$80 + \$10}{2}\right)$ would be equal in value to the instalment loan. The purchaser owed \$80 for 1 month, \$70 for 1 month, and \$10 less each succeeding month. There is a common difference between succeeding numbers in this sequence of numbers, called a number *series*. In any series of this type, the average of the numbers is equal to the average of the first and last terms. In the series below, the average is the sum of the numbers, or 360, divided by 8, or 45.

80 70 60 50 40 30 20 10

This same value can be found by adding the first and last terms of the series and dividing by 2, or $\frac{80 + 10}{2} = 45$. A loan of \$45 for 8 months would be the equivalent of a loan of \$80 for 1 month, \$70 for 1 month, \$60 for 1 month, . . . and \$10 for 1 month. The yearly interest charged for the loan compared with the average amount of the loan, or \$45, would give the rate of interest charged. The solutions of the following three problems show how to find the approximate rate of interest charged on instalment loans.

1. A television set sells for \$240 cash, or \$20 down and \$20 a month for 12 months. What is the interest charge on the instalment loan?

The interest charge (carrying charge) is the difference between the cash price and the instalment price, or \$20.

The initial unpaid balance is \$240 and the last payment is \$20. The average of \$240 and \$20 is \$130. $\$20 \div \$130 = \frac{2}{13}$, or approximately 15%.

[4] A closer approximation of the true rate can be found by using the cash price instead of the instalment price. Then the average amount of an instalment loan would be the average of the cash price less the down payment and the last payment.

2. A watch sells for \$80 cash or \$10 down and \$8 a month for 10 months. Find the interest charge for the privilege of instalment buying.

The interest charge is \$10. A loan equal in value to a loan of \$80 paid off in 10 equal monthly instalments of \$8 is \$44, $\left(\frac{\$80 + \$8}{2}\right)$. The interest charge for 10 months, or $\frac{5}{6}$ year, is \$10. Therefore, the interest charge at that rate for a year would be \$10 $\div \frac{5}{6}$, or \$12. Since the rate of interest means the rate per year, the rate would be found by dividing \$12 by \$44, or approximately 27%.

3. A loan of \$400 for the purchase of a used car was paid off in 18 equal monthly payments of \$25 each. What was the rate of interest on the loan?

The interest charge was \$50 (\$450 − \$400). A loan equal in value to the average of \$400 and \$25 is \$212.50.

The interest charged for 18 months ($1\frac{1}{2}$ years) was \$50. Therefore, the yearly interest would be found by dividing \$50 by $1\frac{1}{2}$ or \$33.33. Then \$33.33 $\div$ \$212.50 is approximately .16 or 16%.

The interest rate charged was approximately 16%.

The method of finding the approximate rate of interest on an instalment loan may be summarized as follows:

1. Find the average of the unpaid balance—instalment price less down payment—and the last regular payment

2. Find the interest charge. This is the difference between the cash price and the instalment price. When the time of payment is not 1 year, express the number of months as a fractional part of 1 year. Then divide the interest charge by the fraction which represents time as a part of a year.

3. Divide the yearly interest as found in statement 2 with the average of the two amounts as found in statement 1 and express the result as a per cent.

The above method should be given as part of the enrichment program for students in the upper grades. Only the superior students are able to understand the steps in the solution. The

approximate rate of interest on an instalment loan can be found by use of different formulas. One of these formulas is $r = \frac{24c}{p\,(n+1)}$ in which c is the interest charge, p is the difference between the cash price and the down payment, and n is the number of payments. This formula should not be used until the student has the background to understand its derivation.

f. Compound Interest

Difference between Simple Interest and Compound Interest

If money is invested at compound interest, the interest when due is added to the principal to form a new principal. Interest is then paid on the new principal. Most students in the junior high school are familiar with the application of compound interest to either a savings account in a bank or to a Savings Bond. Interest on accounts in savings banks usually is compounded either quarterly or semiannually. The period of time from the date when interest on an account is due to the next similar date is called the *conversion period* of compound interest. There are four conversion periods per year when interest is compounded quarterly.

The purchaser of a Savings Bond pays three-fourths of the face value of the bond. Thus, a \$100-bond of this kind costs \$75. After a period of 8 years 11 months, the value of the bond is equal to its face value, or \$100. The \$25 in excess of the purchase price is the accumulation of interest at $3\frac{1}{4}\%$, compounded semiannually.

The student should find the difference between simple and compound interest on a given principal compounded annually for a period of 3 years. The work shows the steps in finding the *amount* of \$100 at 4% interest, compounded annually, for 3 years. Interest is not paid on a fractional part of a dollar.

\$100.00,	original principal
4.00,	interest at end of 1 yr.
\$104.00,	principal at end of 1 yr.
4.16,	interest at end of 2 yr.
\$108.16,	principal at end of 2 yr.
4.32,	interest at end of 3 yr.
\$112.48,	principal at end of 3 yr.

The interest on \$100 at 4%, compounded annually, for 3 years accumulates to \$12.48. At simple interest the accumulation would be \$12, hence the difference between simple interest and compound interest in the given problem is 48¢. From the illustration it is seen that the computation involved in finding compound interest is time consuming. For this reason interest tables have been compiled for finding compound interest. Interest tables show the accumulation of \$1 at different rates for certain periods of time when interest is compounded annually.

The table below shows the amounts of a principal of \$1, interest compounded annually, at different rates for certain periods of time. The student should be able to find the amount of a given principal placed at any of the rates shown for any of the years given.

The difference between simple interest and compound interest can be shown very strikingly by comparing the amounts from two equal principals placed at the same rate for a long period of time if one principal is placed at simple interest and the other principal is at compound interest. History relates that the Dutch

TABLE VIII. AMOUNT OF \$1 COMPOUNDED ANNUALLY

No. of Years	Rate of Interest				
	1%	2%	3%	4%	5%
1	1.0100	1.0200	1.0300	1.0400	1.0500
2	1.0201	1.0404	1.0609	1.0816	1.1025
3	1.0303	1.0612	1.0927	1.1249	1.1576
4	1.0406	1.0824	1.1255	1.1699	1.2155
5	1.0510	1.1041	1.1593	1.2167	1.2763
6	1.0615	1.1262	1.1941	1.2653	1.3401
7	1.0721	1.1487	1.2299	1.3159	1.4071
8	1.0829	1.1717	1.2668	1.3686	1.4775
9	1.0937	1.1951	1 3048	1.4233	1.5513
10	1.1046	1 2190	1.3439	1.4802	1.6289
15	1.1610	1.3459	1.5580	1.8009	2.0789
20	1.2202	1.4859	1.8061	2 1911	2.6533

purchased Manhattan from the Indians in 1626 for $24. If this sum had been invested at 6% simple interest from that date until 1950, a period of 324 years, the amount would have been approximately $490.

Money invested at 6%, if the interest is compounded annually, doubles itself in approximately 12 years. In a period of 324 years, the interest would have doubled itself approximately 27 times, which may be expressed as 2^{27}. The reader should understand that a principal of $24 at 6% interest, compounded annually, would amount to an extremely large sum. The amount would be approximately $3,800,000,000. Such is the magic in compound interest!

Many more uses are made of compound interest than those already mentioned. Compound interest functions in such social institutions as in life insurance, Social Security, and in sinking funds for the liquidation or amortization of public or corporate debts.

Life Insurance

Life insurance is a money protection against loss of income due to loss of life. Payment is made for insurance protection periodically. Such protection is acquired through the cooperative sharing of *risks* of misfortune by groups on the assumption that only a small proportion of the group is likely to experience misfortune at the same time. Members of the group contribute to a single fund. Corporations functioning as insuring bodies compute risks scientifically by various calculations and from experience tables. One of the familiar tables of this kind is the *American Experience Table of Mortality* which gives the life experience of 100,000 people at age 10 until the age of 96 when these people have passed away.

Today most life insurance companies use the *Commissioners Standard Ordinary Mortality Table* (CSO) which is based on the experiences of life insurance companies from 1930–40. This table gives the life experience of 1,000,000 beginning at age 1 through age 99 at which the computations end. Part of this table is given on page 261.

TABLE IX. COMMISSIONERS STANDARD ORDINARY MORTALITY TABLE

Age	Number Living	Deaths Each Year	Deaths per 1000	Years of Expectancy
1	1,000,000	5,770	5.77	62.75
5	983,817	2,715	2.76	58.74
10	971,804	1,914	1.97	55.47
20	951,483	2,312	2.43	46.54
30	924,609	3,292	3.56	37.74
40	883,342	5,459	6.18	29.25
50	810,900	9,990	12.32	21.37
60	677,771	18,022	26.59	14.50
70	454,548	26,955	59.30	8.99
80	181,765	23,966	131.85	5.06
90	21,577	6,063	280.99	2.58
95	3,011	1,193	396.21	1.63
99	125	125	1000.00	.50

The column on the right gives the *life expectancy* at given ages. Life expectancy is the average number of years of life remaining at any given age. Thus, at age 1, a person, on the average, should live to be approximately 64 years of age, but at age 30, he should live to be approximately 68 years of age.

The table shows that the death rate per 1000 population is lowest at about 10 years of age and then it increases slowly until age 40. From that point the rate increases rapidly until it reaches a rate of 100 per cent at age 99. The death rate per 1000 population is found by dividing the number of deaths for any given year by the number of thousands of people living at the beginning of that year. For example, at age 50 the average number of deaths is 9990. At the beginning of the year there were 810,900 people living. Hence, 9990 ÷ 810.9 will give the death rate per 1000, or 12.32.

From the table it is possible to determine the cost of the risk of death at a given age. If a company insures the lives of 1000 persons at age 50, it would provide a sum of $12,320 (12.32 × $1000) for the payment of death claims during the year, assuming that each of the insured had a policy of $1000. This death claim

is equal to an average of $12.32 for each of the insured, who would pay this sum, known as the *mortality cost* of the insurance for that year.

The Yearly Premium

In order to understand insurance, it is necessary to know what constitutes the *premium*. The premium is the sum the insured pays the company for the *policy* or contract between the insured and the company. In life insurance the premium is generally computed on a yearly basis. The three elements of the premium of life insurance are the *mortality*, *loading*, and *reserve*. We have found that the mortality cost is determined from a table of the type given on page 261.

The loading expenses are those costs which result from the operation of the company. Such items as taxes, agents' commissions, and other expenses common to the operation of a large business are part of the loading expenses of an insurance company.

The reserve of the premium is the sum of money that is set aside each year at compound interest to equal the face of the policy at maturity. If at age 30 the insured has a policy which will mature in 20 years, enough money each year must be set aside at $2\frac{1}{2}\%$ or 3%, compound interest, to equal the face of the policy, or $1000, when the insured becomes 50 years of age. If the insured surrenders his policy before it matures, he receives a refund from the company of the premiums paid which were assigned to reserve plus the accumulation of the reserve at compound interest. This amount is called the *cash surrender value* of the policy.

The cost of the three items in the premium varies slightly among different insurance companies. Most companies use the same table of mortality and compute the reserve at approximately the same rate of interest, hence the greatest variation in premiums would result from the cost of loading. Approximately 16 cents of the premium dollar goes to this item and the remaining 84 cents is divided proportionately between mortality cost and reserve.

Kinds of Policies

There are many kinds of life insurance policies. Three types of standard policies are: (1) *straight or ordinary life*; (2) *limited payment life*; and (3) *endowment*.

A straight life policy matures only at the death of the insured or when the insured reaches the terminal age of the mortality table. The owner of the policy pays the annual premium for the duration of the policy. A policy of this type affords protection at a cheaper rate than the other two types of policies as shown by the table below.

The owner of a limited payment policy pays premiums for a designated number of years, such as 25 years, and then the policy is paid up, but the policy does not mature until the death of the insured. A policy of this kind usually is recommended for a person under 40 years of age. He can pay the premiums while he is able to earn an income before he retires. At his retirement the policy would be paid up.

An endowment policy matures either at the death of the insured or at the end of a specified number of years, as 20 years. It should be understood that all life insurance policies carried by a person mature at his death. The distinguishing feature of an endowment policy is its emphasis on savings as well as its provision for protection. Because of its emphasis on savings, the cost of the reserve makes the premium higher for an endowment policy than for either of the other two policies as shown below.

TABLE X. YEARLY PREMIUMS PER $1000 FOR DIFFERENT KINDS OF POLICIES

Age of Insured	Kind of Policy: Straight Life	20-Payment Life	20-Year Endowment
20	$17.60	$29.08	$47.99
25	20.24	32.03	48.72
30	23.51	35.42	49.78
35	27.61	39.36	51.33
40	32.82	44.06	53.62

At a recent date a large insurance company used the rates given in the table for yearly premiums at different age levels for the three types of policies mentioned. The data showed that as the age of the insured increased the difference in the premiums of the three policies decreased.

A fourth type of policy is called *term insurance*. Term insurance usually is carried for a limited period of time, such as 5 years. No savings feature is considered in term insurance, hence it is issued for protection only. The premium consists of mortality cost and loading expenses. There is no cost for reserve. Therefore, the premium for term insurance is much less than the premium of the standard policies discussed above.

The teacher will find the following filmstrips effective instructional aids for students studying life insurance in the upper grades:

How Life Insurance Operates
How Life Insurance Policies Work

These filmstrips may be secured from the Institute of Life Insurance, 488 Madison Avenue, New York City.

Fire Insurance

Fire insurance is one of the most familiar kinds of insurance. As the name designates, fire insurance is a money protection against the loss of property by fire. The student should understand the elements which constitute the premium for this kind of insurance. The two factors affecting the premium are *risk* and *loading*. There is no provision for reserve in any kind of insurance except for life insurance.

The risk involved in fire insurance on a building depends upon the following:

1. The kind of structure
2. The location of the structure
3. The use of the structure
4. The surroundings.

It is apparent that a stone or brick structure with a fire resistant roof should have a lower rate of fire insurance than a

frame structure with a wooden shingle roof. The rate of insurance on a structure should be lower in a city that has good fire protection than in a rural area without such protection. A one-family house should have a lower rate of insurance than a building of the same structure which may house five or more families. A house located in a residential area should have a lower rate of fire insurance than a house of the same structure located near a factory making combustibles.

It is possible to make a savings in the premium on a fire insurance policy by paying the premium for periods longer than one year. The rates for different periods of time are as follows:

Period	*Rate*
2 years	$1\frac{3}{4}$ times the rate for 1 year
3 years	$2\frac{1}{2}$ times the rate for 1 year
4 years	$3\frac{1}{4}$ times the rate for 1 year
5 years	4 times the rate for 1 year

It is possible to make a savings of 20 per cent of the yearly rate by purchasing a fire insurance policy for 5 years instead of five yearly policies. Very superior students in a class should be able to compute the per cent of savings possible for each of the periods given in the table.

Casualty Insurance

Casualty insurance consists of insurance covering losses due to accidents, storms, or similar occurrences. *Liability insurance* for automobiles is one of the familiar kinds of casualty insurance. The student should understand that the premium for casualty insurance depends upon *risk* and *loading expenses*. The risk involved in casualty insurance for a car depends upon such factors as the residence of the insured, his age, and the use made of the car.

The student should understand that the risk of total liability of the face of the policy decreases as the face of the policy increases. A familiar form of this policy is one for \$5000–\$10,000–\$5000. This means that the company issuing the insurance

assumes liability in case of personal injury by the car of the insured to the extent of $5000 for one person, $10,000 for two or more persons, and $5000 for property damage. In certain parts of this country, the cost of this insurance on a sedan would be approximately $60. To double the liability insurance so that the coverage would be $10,000–$20,000 with the coverage for property the same, the extra premium would be approximately $3. For about $2 more ($65), the liability insurance could be increased to $25,000–$50,000. These data show that the chance of the company's paying more than $5000 to one person for injury is small.

In teaching such topics as fire or casualty insurance, the teacher should stress two things: (1) what the elements which affect the premium are and (2) how safety and care can reduce the risk involved. As the risk is reduced, the premium also will be reduced.

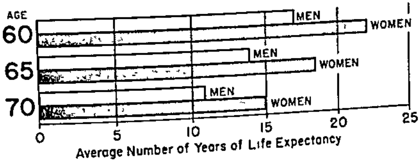
Annuities

An *annuity* provides a specific income for the life of the person, called the *annuitant*, who holds the policy. For the payment of a given sum, such as $1000, to an insurance company, the annuitant will receive a specified income for life. A person at age 40 may deposit $1000 and in return the insurance company would give him a guaranteed income for life beginning at any specified date. The annuitant may decide that he would prefer to receive an income beginning at age 60. During the intervening 20 years the $1000 would be drawing compound interest so as to make the amount to be paid the annuitant more than the principal. Naturally, the yearly income would be greater for the annuitant at age 65 than at age 60.

An annuity is the opposite of life insurance. Life insurance is to build an estate, but an annuity is to liquidate an estate. Life insurance is to give financial protection to dependents, but an annuity is to give financial protection to the annuitant.

The graph shows the average life expectancy of men and women at ages 60, 65, and 70 according to the records of one large insurance company. From the graph it is seen that, on the

ANNUITANT LIFE EXPECTANCY



average, women live approximately 4 years longer than men. Therefore, an annuity to provide a specified income at any given age, as at 60, would cost a woman more than a man.

Because of individual variations in life span, a person who wishes to live off investments dares not dip very deeply into the principal lest he outlive it. If he uses income alone he needs a capital sum of $40,000 at 3 per cent interest to provide $1200 a year. By sharing income security with thousands of other persons through annuities, a man can be assured of an income regardless of whether he lives one or forty years. In this case, since both principal and interest are used, the capital sum needed at age 65 to provide $1200 a year for life is about $15,500 for a man and $18,500 for a woman.

The teachers of many states are members of retirement funds. Frequently, the teacher and the state make an equal contribution to these funds. The teacher, with the aid of the state, buys an annuity so that upon retirement he will receive a given income for the rest of his life. The most familiar illustration of an annuity is found in *Social Security*.

Social Security

Most wage earners today belong to Social Security. In the junior high school many students who have part time jobs in certain occupations are members of Social Security. A certain per cent of the worker's wage is deducted for this governmental agency and the employer matches the contribution of the employee.

The government specifies the maximum amount of yearly wage that is subject to income deduction for Social Security. At a recent date this wage was \$4200. The wage earner contributes to Social Security until he is 65 years of age, at which time he is assured of an income for life. The yearly income from this source depends upon the number of years he has contributed to the fund and also upon his yearly wage during his period of contribution.

Questions, Problems, and Topics for Discussion

1. Name some exploratory materials which could be used by a student to help him understand the meaning of per cent.

2. Show some kind of activity in which the student should participate in order for him to discover the method to use for finding a per cent of a number.

3. For most students, what is the most difficult part of the solution of a problem involving finding what per cent one number is of another number? What are some of the things which may be done to help the student overcome this difficulty?

4. How could you have a student discover that it is possible to have an increase of 100% in the cost of an article, but that there cannot be a reduction of 100% in the cost of an article which is sold?

5. Show why you would or would not find a fractional part of one per cent by expressing the per cent as a three- or more-place decimal and then multiplying the given amount by that decimal.

6. Write three problems which illustrate the three kinds of problems for finding a number when a per cent of it is given.

7. Write three problems to illustrate the three usages of per cent. Indicate which usage each problem represents.

8. What is meant by spacing a topic in the curriculum? Indicate how the topic of per cent should be spaced in the curriculum.

9. What is the percentage formula? Under what conditions should this formula be used in teaching the topic of per cent?

10. Show why an increase of 10% in wages and then a 10% decrease in wages would give a net wage less than the net wage before there was a change in the wages.

11. Make a list of some of the chief social applications of per cents.

12. A *small loan* is sometimes defined as a loan of \$500 or less. In a certain state a small loan company is permitted to charge $2\frac{1}{2}\%$ a month on the first \$300 of a loan and $\frac{1}{2}\%$ a month on the part of a loan in excess of \$300. Using these rates, find the monthly interest on a loan of \$450 [Answer, \$8.25]

13. Make a list of the concepts used in merchandising and define each of these terms.

14. Make a list of three advantages and three disadvantages of instalment buying.

15. A musical instrument sells for $150 cash or $25 down and $10 a month for 15 months. Find the approximate rate of interest charged for the privilege of instalment buying. [Answer, 25%]

16. A used car sells for $600 cash, or $200 down and $50 a month for 9 months. Find the approximate rate of interest charged for the privilege of instalment buying. [Answer, 21%]

17. What are the elements in the premiums for life insurance, fire insurance, and casualty insurance?

18. Why should the owner of a house pay a different rate of fire insurance on his household goods than on the house? Which would be more?

19. A man wrote to an insurance company as follows "You have insured my car for 15 years and I have not had an accident I should be given a refund of part of my premiums." Do you think the company gave him the refund? Should he be given the refund?

20. Find the following data about Social Security. **a.** The rate of deductions assigned to the fund **b.** The maximum wage subject to deduction **c.** The minimum number of quarters one must be a member in order to receive benefits at age 65 **d.** The maximum benefit a participant may receive monthly from this source.

Suggested Readings

Brueckner, Leo J. and Grossnickle, Foster E. *Making Arithmetic Meaningful*, pp. 426–429. Philadelphia: The John C. Winston Co., 1953

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Clark, John R. and Eads, Laura *Guiding Arithmetic Learning*, pp 109–204 Yonkers, N. Y.: World Book Co., 1954.

Huebner, Solomon S. *Economics of Life Insurance, Human Life Values: Their Financial Organization, Management, and Liquidation*, p 272 New York Appleton-Century, 1944.

Shaw, George B. "The Vice of Gambling and the Virtue of Insurance," *The World of Mathematics*, Vol 3, pp 1524–1531, edited by James R Newman. New York: Simon and Schuster, 1956.

Using Consumer Credit, p. 107. Consumer Education Series, Unit No 7 National Association of Secondary-School Principals Washington, D. C.: National Education Association, 1947

Chapter 8

Instructional Units in Mathematics

IN preceding chapters it has been shown that learning is a growth process, that learning proceeds best when it takes place in social situations that are meaningful to the learners, and that learning proceeds best when the learners understand what they are learning and it makes sense to them. They should be given the opportunity to discover meanings and relationships and to make and apply generalizations. Methods and materials of instruction must be adapted to the level of ability and development of the learner. In the selection of content and activities, special consideration should be given to the differences in the needs, interests, and background of experiences of the students.

In this chapter the following topics are discussed:

a. The nature of instructional units
b. Building units
c. Subject-matter units
d. Enriching learning through experience units
e. Illustrative experience units.

a. The Nature of Instructional Units

Points of View as to the Nature of Units

In current literature on methodology we find numerous expressions that are used to identify different kinds of units of instruction. The most important of these are subject-matter units,

process units, experience units, and enterprises. The traditional daily page by page assignment based on a textbook is completely out of line with modern conceptions of how learning takes place. As Burton[1] has said, "It would be difficult to devise an educational practice so grossly ineffective, so certainly calculated to interfere with learning, as a page assignment to a single text followed by a formal quiz." Yet this practice is characteristic of instruction in many classrooms.

Subject-Matter Units

Subject-matter units are subject matter centered. They are arranged around a central core within the subject matter itself. The core may be some mathematical operation, a topic, a theme, or a center of interest. Illustrations of subject-matter units are:

1. How to divide by decimals
2. How to compute interest
3. The history of our number system
4. How to make and interpret graphs.

Process Units

Process units focus on the development of effective patterns and habits of thinking, that is, thought processes. Smith[2] has identified three of the most important thought processes used in daily life which are basically different, one from the other, in their nature.

However, in spite of the desirability of developing instructional units that emphasize thought processes of importance, little research has been undertaken along this line in mathematics. It is important that an analysis be made of the possible ways in which instruction in mathematics can contribute to effective ways of thinking in social situations. Interest in this problem is shown by current emphasis on meanings and understandings as

[1] Burton, W. H. *Guidance of Learning Activities*, page 337. New York: Appleton-Century-Crofts, Inc., 1952. Chapters 9 to 13 contain an excellent theoretical discussion of the procedures to follow in the development of various kinds of units.

[2] Smith, B. O., Stanley, W., and Shores, J. H. *Fundamentals of Curriculum Development*, Chapter 23. Yonkers, N. Y.: World Book Co., 1950.

well as on the improvement of problem solving. Routine drill on number operations is of no value as such in the improvement of significant thought processes.

Enterprises and Organized School Experiences

Enterprises are undertakings, usually outside the classroom, through which students come into direct contact with quantitative aspects of social institutions similar to those found in life outside the school and participate in their activities. Usually they are a part of the organized life of the school. Illustrations of enterprises are the following:

1. Operating the school savings bank
2. Establishing a school credit union
3. Planning the school activity budget
4. Laying out a school playground or athletic field
5. Making a survey of fire hazards in the school and community
6. Participation in a bond drive to raise funds for a new school building.

A valuable form of learning experience is found in the schools of Winnetka, Illinois. There the teachers have encouraged the formation by students of three different kinds of simple corporate school-centered organizations — public, private, and cooperative.[3] An example of a public organization is a mutual insurance company which insures the students against accidental breakage of dishes in the lunchroom. An example of private corporation is a school credit union organized and incorporated under the laws of the school community to encourage savings, to provide small loans, and to afford experience in the control of credit. An example of a cooperative organization is a company that manufactures and distributes school supplies and operates an exchange of such articles and divides profits among shareholders. Evidently the faculty in Winnetka is convinced that such experiences have real educative value. Other organizations similar in nature are established from time to time as the possibility of valuable

[3] Logan, S. R. "Adventuring with Little Corporations," *The Clearing House*. 20 73-81, 21 201-213.

educative experience arises. "Junior Achievement" is the best known example of group participation by youth in the actual conduct of productive enterprises. 4H clubs are another.

Experience Units

Experience units consist of a series of educative experiences organized around some pupil problem, need, or purpose. Socially useful subject-matter and materials are utilized in the activities undertaken by the group to solve the problem faced, to achieve a purpose, or to satisfy a need. The learning outcomes inherent in the process of experiencing are: reflective thinking, functional use of the basic tools of learning, practice in cooperative group action, and the development of a variety of insights, interests, attitudes, and appreciations. Illustrations of experience units are the following:

1. How is arithmetic used in the games that people play and in their sports?
2. What is the cost of operating an automobile?
3. How do banks help in and serve the community?
4. In what ways are people paid for the work they do?

In experience units all students regardless of the level of their mathematical development or intelligence work together as a group, each one contributing to the activity under way in terms of his interests, abilities, and special talents and aptitudes. This approach is a very valuable way of exploring and developing the many aspects of the personality of the individual student

Common Elements in the Various Kinds of Units

Careful study will show that actually there is a great deal of overlap among these four kinds of units. Subject matter is involved in all types, but its mastery is stressed in subject-matter units. Thought processes are likewise involved in all types of units but special consideration is given to the development of particular kinds of thought processes in process units. It is obvious that subject matter is involved in process units. In experience units and in so-called enterprises the stress is on methods of

learning and kinds of social behavior, although subject matter and thought processes necessarily operate in these types of instructional units.

In the ideal instructional unit there is a combination of subject-matter learning and of thought processes used in problem solving, critical evaluation, and generalizing which will be of real value to the learner and aid him in integrating his learning. Integration is aided when the learner sees sense and value in what he is studying and in the thought and behavior patterns that are stressed within his learning activities.

These experiences may be either direct or vicarious. Little is gained by attempting to classify any given unit under some particular category. Actually the title of an instructional unit cannot be depended on to identify the kind of unit it is. A unit can be classified as to type only by observing what takes place in the classroom in the course of its development.

Both textbooks and direct experience have a place side by side in the modern mathematics program. Modern textbooks try to relate mathematics to life experiences and to suggest aspects of community life to explore to socialize and vitalize instruction. Teachers should make use of the suggestions given in textbooks and supplement them freely with their own ideas. In this way interest is likely to be aroused and maintained, and purposes emerge so that the learner will go on from there himself.

In the following discussion we shall illustrate two kinds of instructional units, one that focuses on the systematic teaching of the technical subject matter of mathematics, the other that utilizes primarily the experience approach.

b. Building Units

Various Ways of Building Units

Units may be built in a number of different ways. The most valuable type of unit in terms of the richness of its possible outcomes is one that is chosen for study cooperatively by the teachers and pupils on the basis of the needs, interests, and purposes of the students. The pupils set their own goals, participate in the

planning of the activities, and evaluate the results of their endeavors. These units may grow out of class discussions, out of problems faced by the group, or out of situations in the community. They may center around such topics as the mathematics of aviation, how discounts help to reduce costs, or the mathematics of sports.

Units can be developed in terms of a flexible planned pattern of topics to be dealt with at successive grade levels, and the content adjusted to the level of mathematics the pupils have studied, for example, "How banks serve the community."

Units can be chosen from among those that may be available in courses of study, resource units, and other published materials. Resource units often supply a great variety of material from which suitable topics and content can be selected, especially when they have been prepared with local situations specifically in mind.

In modern textbooks the contents are often organized as units. However, because of their general nature, the teacher finds it necessary to supplement these units so as to bring into the unit local illustrations and situations related to the topic being discussed. In some textbooks suggestions are given for local research by students on matters being discussed. Enrichment can thus be furnished as a means of providing for the more able students and for creating interest on the part of all in the class.

At all times the teacher should feel free to discuss with the students events of current interest, for example, insurance as related to a fire in the neighborhood.

The systematic review of work with number operations can also be organized in units according to process or numbers used. These could contain diagnostic inventory tests on the various processes, and necessary reteaching materials and practice exercises. The approach should be such as to give students a *new* view of topics previously studied rather than a mere *re*-view.

Items to be Considered in Selecting and Building Units

In selecting and building a unit the teacher should give careful consideration to the points listed on the following page.

1. The readiness of the students for the topic and the level of development of their mathematical skills

2. The setting up of the objectives

3. Possible ways of approach to be used in introducing the unit

4. An outline of the subject-matter content

5. Availability of necessary materials, such as reference sources, pamphlets, visual aids, displays, apparatus, and devices

6. Possible use of community resources

7. Provisions for a variety of pupil activity, including problem solving, constructions, excursions, creative work, field work, experiments, and practice materials

8. Methods of evaluating the outcomes of the work, of diagnosing difficulties, and remedying deficiencies.

Mathematical Themes Basic in Instructional Units

The authors have found that their students recognize the value of some sort of guide or checklist to assist them in the selection of the contents of subject-matter units. The following list of twelve basic themes is very helpful as a basis for enriching and extending the contents of all types of instructional units. Six of the themes are related directly to the mathematical phase of the subject and six of them to the social phase.

Themes Basic in Unit Construction

1. Themes stressing the mathematical phase of arithmetic
   a. Arithmetic is based on a *number system* invented by man and developed by him to a high state of perfection, which enables him to deal in an orderly way with quantitative aspects of the environment.
   b. To reduce the labor involved in counting, man first invented *methods of computation* and, more recently, highly efficient mechanical computing devices.
   c. Man has devised units of measurement which enable him to deal with quantitative aspects of the environment in a *precise* way.

d. Mathematical language affords "an exact and easily workable *symbolism* for the expression of ideas" in a definite, precise way.

e. By means of number and standard units of measurement man can analyze and array *systematically* facts dealing with quantitative aspects of the environment.

f. All intellectual achievement is based on the ability to see *relationships between variables* which, when clearly and accurately presented by mathematical techniques, can be better understood.

2. Themes stressing the social phase of arithmetic

a. The use of quantitative methods and devices has enabled man to gain increasing *control over nature* and to direct its forces to his own ends.

b. Present-day quantitative methods and devices are the more or less perfected end products of an *evolving* group of social institutions.

c. As new needs arise in various localities, often simultaneously, man uses his intelligence and *invents* new quantitative techniques for dealing with them; science evaluates these varied methods and finally efficient standard procedures are evolved.

d. The efficiency of number and quantitative methods has been largely instrumental in the development of *inter-cooperation* among individuals, states, and nations of the earth.

e. Intelligent and effective production, distribution, and consumption of goods depend on the availability of accurate, reliable information and the disposition of the individuals concerned to base action on these facts. *Economic competence* is important.

f. Lack of precise accurate information enables any individual to mark the boundary of his ignorance in any field; mathematical techniques enable him to estimate the *extent of possible error* in any information he may have.

If these themes are borne in mind in building and developing units, the work of the students will be quite likely to deal with all important mathematical aspects of the topic.

c. Subject-Matter Units

Units for Teaching Technical Subject Matter

The subject matter of mathematics should be presented in relatively large units of related elements, for instance, division by decimals. In accordance with the "growth" principle the teacher should break down the process into a series of carefully graded sub-types beginning with the simplest combination of skills and ending with the most complex type of example that is to be taught. The sub-units in a unit for per cent in Grade 7 level may be listed as follows:

1. The meaning of per cent developed through manipulation and visual activities
2. Changing fractions to per cents (easy types)
3. Finding a per cent of a number
4. Expressing hundredths as fractions, decimals, and per cents
5. Finding a per cent of a number, using fractional equivalent
6. Finding what per cent one number is of another (the per cent is a whole number)
7. Per cents greater than 100
8. Finding per cents of reduction and change
9. Rounding off per cents.

Usually the textbook in use will provide the necessary instructional materials and practice exercises, except for concrete materials that the teacher must supply. Instructional procedures should emphasize the social utility of what is to be learned and basic meanings and understandings as described in Chapter 3. The teacher should conduct the work in such a way that the students will do the reflective thinking that will lead them to discover meanings and relationships and arrive at basic principles of procedure.

Steps in Teaching a New Step in a Process

The sequence to be followed in teaching any new work in a number operation so as to make it mathematically meaningful

and socially significant to the class as a whole may be briefly stated as follows:

1. Present the simplest step in the new operation in a concrete social situation in which the need of the new step arises so that its social use will be evident to the learners. The need may have arisen in some class activity or in connection with the work in some other curriculum area, such as science or music, or be a logical next step in an instructional unit.

2. Have the pupils indicate the operation that is involved and write it as an example. Have them point out what the new difficulty is and how the example differs from examples they have already learned to work in the same process.

3. Have the pupils try to discover the answer by drawing on past experience or by devising original ways of working the example. The purpose here is to give meaning to the step. Very often procedures will be given by the pupils that approximate closely the algorism to be taught. This indicates a high degree of readiness for the new work.

4. Develop now in a meaningful way the algorism to be learned so that the children will understand the mathematical meaning of the steps involved. Use concrete materials, pictures, diagrams, and visual representations to visualize the meaning of the various steps in the solution to supplement the background given in the text. The modern textbook usually provides the essential background and explanations. Often a visual representation is included.

5. Have the pupils go over three or four illustrative worked-out models of similar examples and explain the procedure used. Also have each of the examples worked at the chalkboard by pupils so that all can see the steps in the solution Show them how to check the answer.

6. Then have all of the pupils copy and work the model examples at their seats without referring to the worked-out models which may be concealed by a wall map if on the chalkboard. The pupils who can work the examples correctly and check the answers are ready for practice to develop skill and proficiency. Follow the slogan: Never assign practice until the pupils understand what they are to practice.

7. Reteach the steps to those pupils who were not able to work the model examples correctly. They are likely to need more concrete work than was given originally.

8. On subsequent days give the pupils the opportunity to apply the new step in a variety of ways in social situations. The wider the variety of applications, the more likely it is that the new step will be mastered. Give additional practice also to increase skill and proficiency. Try to have pupils who may be using inefficient procedures learn to use increasingly mature methods of thinking as they proceed.

Control and proficiency of skills should be developed through use in social situations, practice in a variety of contexts, and repetitive drill. Generalizations and relationships with other processes should be developed to aid the learner to organize what he is studying and to integrate it with what he already has learned.

Subject-Matter Units Dealing with Social Applications of Mathematics

A similar procedure should be followed whenever the textbook is the main source of subject matter in the study of such technical mathematical topics as applications of per cents, intuitive geometry, and elements of algebra. In connection with the study of such topics as insurance, taxation, banking, and the like the teacher should emphasize critical reflective thinking about these subjects and assist the students to evaluate current practices. Of course the teacher may choose to limit the work to the study of brief explanations and discussions in the textbook and the solution of verbal problems that usually appear in them. Such a procedure is of limited value as a learning experience. The content may have little appeal to the students and not be related to their experiences, may be sketchy and not well organized, and perhaps contain concepts that they are not able to grasp.

Two illustrative outlines of subject matter suitable for instructional units are given below. The first of these units, Our Number System, will require the students to consult various sources for the needed information, an excellent means of

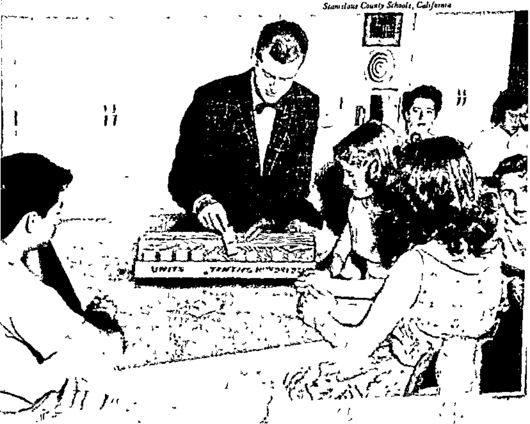
developing study skills and the ability to read and organize mathematical materials. The second unit is an adaptation of a series of sub-units that appear in a chapter of a seventh grade textbook dealing with the topic, Mathematics of the Home.

I. Our Number System

1. Early devices used in counting and grouping
2. Systems of notation – past and present
3. Principles underlying our number system, including place value, symbols used to express values, the base of 10, and 0 as place holder
4. The operation of the number system in number processes, as in carrying in addition or regrouping in subtraction
5. Various functions and special meanings of zero
6. Common fractions an addition to our number system, and their various meanings and uses
7. Decimal fractions – a recent extension of our number system involving the use of places to the right of one's place

Place-value charts can be made of wood as well as folded oaktag.

Stanislaus County Schools, California



8. Approximation and the rounding off of numbers
9. The number scale and rational numbers
10. Positive and negative numbers – a further extension of our number system
11. Radical expressions – square root
12. Other number systems and systems of notation, including scientific notation through use of exponents, as, 1 light year = 9.46×10^{12} Km.

II. Mathematics of the Home
1. Planning the family budget
2. The cost of the food we eat
3. Keeping a cash account
4. Cost of operating a car
5. The farmer's share of the food dollar
6. How we measure food value
7. It's cheaper to buy in quantity
8. Finding the cost of the electricity used to operate appliances
9. Finding the cost of water consumption
10. Finding the cost of various ways of heating the home.

d. Enriching Learning through Experience Units

Steps in the Development of Experience Units

When the study of mathematics deals with problems that are related to the needs of the students and topics that they recognize are of value, the instructional program will be considerably different in nature from the procedures just described. Emphasis is then placed on learning through experiences in which not only the specific mathematical skills and concepts are involved but also their social applications. The sequence of steps[4] that are involved in the selection and development of experience units may be stated briefly as on the following pages.

[4] An excellent analysis of the development of enriched instructional units is given in Burton, W. H. *Guidance of Learning Activities*, Chapters 13 and 14. New York: Appleton-Century-Crofts, Inc , 1952.

1. The teacher first systematically studies the needs, interests, and stages of development of the pupils so as to identify socially significant problems or topics, the consideration of which will in the teacher's opinion be likely to contribute to the well-rounded growth of the students and make possible a wide variety of learning experiences.

2. The attention of the class then is brought sharply to a focus on the selected area. This may be accomplished by considering with them some need or problem that has arisen which is of concern to the pupils. The topic may also grow out of some situation arranged by the teacher to bring the subject before the children. The problem and its significance are clarified by discussion and preliminary exploration. The issues and topics to be considered are then tentatively listed as they are formulated by the pupils or as they grow out of the teacher's suggestions. As the unit develops, new problems arise that are then added to the list.

3. Plans are next set up through group discussion for securing any information that may be needed. The activities selected may involve investigation, research, extensive reading, experimentation, interviewing, construction work, and similar procedures, conducted either by individuals or by groups. Special attention is given by the teacher to the wide range of differences among the pupils so that each one has responsibilities that are interesting to him and that at the same time present a challenge.

4. The desired information then is gathered on a cooperative basis. The teacher tries to direct all activities along productive lines through careful stimulation and guidance. At this stage the progress being made by the various pupils is checked each day by brief reports to the class, and standards are developed against which to check the work. A wealth of related reading matter is made available. When necessary to clarify meanings, experiences are provided which bring the pupils into direct contact with community resources through excursions and field trips. Use may also be made of visual aids of various kinds, such as motion pictures, slides, exhibits, pictures, and drawings. Local illustrative materials are brought in to add concreteness to the work.

5. The information gathered and the results of any activities engaged in by the pupils are then organized in preparation for class discussion. The presentation may be made in the form of oral or written reports, exhibits, constructions, dramatizations, works of art, models, and various other products of creative activity. Evaluation of the product is done continuously by checking contributions against emerging standards developed by teacher and pupils together.

6. The results of study and investigation are then presented to the class, considered by the pupils, and necessary conclusions drawn. Plans for any future action to be taken as a result of the work done by the pupils are then discussed.

7. The outcomes of the unit are finally evaluated by the pupils in terms of the goals achieved and by the teacher in terms of major educational objectives.

The steps listed above should be regarded as general guides only for teaching through experience units and should not be followed as a pattern. The activities carried on by any class should develop naturally as the work progresses and they will likely differ in many respects from the activities of any other class.

e. Illustrative Experience Units

Units of experience of the most valuable types grow out of the cooperative organization by teacher and students of a plan of attack on some problem of social significance that is vital to the students if in the opinion of the teacher its consideration will likely lead to valuable kinds of individual and group learnings. The possible outcomes of these units include not only the learning of a body of important information about the matter being considered, but also a wide variety of wholesome interests, attitudes, appreciations, social insights, and understandings. These units also lead to the development of good study habits and help the pupils to devise methods of attacking future problems. The experience of working together with others in the study and solution of vital problems and of accepting responsibility for assignments by the group contributes to the development of

social qualities and abilities that are universally regarded as fundamental to life in a democratic society.

Within these units there is such a wide variety of activities possible that all of the students can find ways in which each of them can make worthwhile contributions to the group according to his interests, abilities, and special talents. An analysis of the major kinds of activities that may take place in units of experience is given below:

1. Problem solving activities involving
   a. The formulation of a problem
   b. Consideration of the scope and significance of the problem
   c. Planning a method of attack
   d. The assignment of tasks to individuals or groups
   e. The location and gathering of necessary information from persons and printed sources
   f. Research and experimentation needed to get new data
   g. The assembling, organizing, and presenting of findings
   h. Drawing conclusions and making decisions
   i. Taking steps to carry out decisions.
2. Construction activities, such as
   a. Making graphs, charts, and diagrams
   b. Making working models of various kinds, designs
   c. Preparing exhibits, displays
   d. Carrying on experiments
   e. Constructing equipment, tools, utensils
   f. Participating in surveys, drives, campaigns
   g. Buying, selling, building, collecting, etc.
   h. Exploring meanings by manipulating and grouping objective materials.
3. Appreciation activities, such as
   a. Viewing slides, films, pictures
   b. Reading stories, books, articles, bulletins
   c. Hearing radio programs
   d. Viewing plays, dramatizations, performances
   e. Hearing talks by local people, experts
   f. Looking at exhibits, displays.

4. Creative activities, such as
   a. Painting, drawing
   b. Making murals, decorating
   c. Writing original imaginative materials
   d. Planning improvements
   e. Suggesting original novel solutions
   f. Inventing new methods, means, and materials
   g. Dramatizing, performing.
5. Excursions, to such places as
   a. Places of business, banks, stores
   b. Industries, factories, warehouses
   c. Farms, dairies, orchards, etc.
   d. Transportation centers and facilities
   e. Libraries, museums, art centers
   f. Historical spots
   g. Hospitals, medical centers
   h. Housing projects, slum areas
   i. Governmental buildings, post office.
6. Practice activities, such as
   a. Using reading skills
   b. Using language skills, both oral and written
   c. Using number and computational skills
   d. Locating and using sources of information
   e. Using tools, equipment, apparatus
   f. Applying algebraic and geometric concepts.

The activities to be carried on in connection with any unit of experience will of course depend on the nature of the problem or topic being considered, the steps necessary to deal with it fully, the experiences that the children should have to make it meaningful to them, the richness and variety of the instructional materials available, and the accessibility of places and institutions in the community for getting first-hand direct contacts.

Outline of an Experience Unit

The following outline suggests the essential elements of a unit that should be included in a plan for the work of a class.

Generalized Outline of a Proposed Experience Unit

1. *Title of the unit, stated clearly and in such a way as to create interest.*
2. Overview of the unit.
   A general *statement of the* purpose and scope of the unit.
3. Objectives to be achieved.
4. Outline of subject matter to be included in the unit.
5. Possible approaches to the unit.
   (Several should be given)
6. Probable *activities to be undertaken during the unit.*
   a. Problem solving activities
   b. *Concrete and constructive experiences*
   c. Appreciative experiences
   d. Creative experiences
   e. Excursions
   f. Practice experiences
   g. Experiences in democratic living, other than those given above.
7. *Possible culminating activities.*
8. Methods of evaluating outcomes.
   (*At least three specific procedures should be illustrated in* connection with this unit.)
9. Materials and community resources *to be used.*
   a. Materials, objects, etc.
   b. Community resources.
10. Bibliography.

An excellent illustration of an instructional unit emphasizing learning through *experience* is the activities of a group of junior high school students who made a study of *methods of reducing the expenses of the home*, an excellent application of what they had learned about per cent. The following overview of the unit describes *the work of the class:*

1. Overview

This unit grew out of a discussion of the meaning of a 'Thought for the Day' written on the blackboard. The quotation was from *the works of John* Wesley, 'Earn all you can; save all you can; give all you can.' At first the pupils, some of whom came from

very poor homes, saw little connection between saving and giving. Gradually the idea emerged that saving can also mean spending so as to get one's money's worth. The idea was also expressed that, with careful planning and thrifty buying, money can actually be saved no matter how little is earned.

The class first dealt briefly with the question, How will planning help us to spend wisely? The pupils agreed to discuss with their parents the expenses of the home and the spending plans. They also discussed ways in which they earned money and how they spent it. Then, because of the many problems in the homes revealed by this study, it was decided to investigate methods of reducing the major expenses of the home; namely, operating expenses, including heat, light, water, and telephone; necessary expenditures for food, clothing, and shelter; and miscellaneous items, including medical care, insurance, allowances for children, charity, amusements, and reading matter.

Groups of children selected topics of special interest for study and reporting. The investigations needed to secure the desired information required the use of a wide variety of procedures, including reading, interviewing, visitations, discussions with authorities in several fields, and actual experimentation in several instances. Illustrative materials were gathered from the community. Local business practices were discussed and evaluated. The cooperation of several special departments of the school was secured in dealing with a number of the problems, including consultation with teachers of home economics, physical education, social studies, science, and commercial education. The planning in connection with the preparation of the report was done by the groups of pupils concerned, with the assistance of the arithmetic teacher.

The problems discussed involved many applications of percentage—the major topic for the time of year in the mathematics course of study. Frequent use was made of graphs, charts, tables, and diagrams in presenting the information which had been gathered. This last was another important topic in the course of study.

The outcomes of this unit were varied and valuable. The pupils gained a clear conception of the cost of maintaining a home and of ways in which they could assist their parents in reducing expenses. The work done on these problems made the work in percentage meaningful and realistic to the pupils. Through group cooperative procedures they also had valuable experience in the study of concrete problems of concern to them.[5]

[5] *A Guide for Instruction in Arithmetic*, pp 97–98 Curriculum Bulletin No. 3. St. Paul, Minn.: State of Minnesota Department of Education, 1948.

The subject matter included was as follows:

2. Subject Matter
   a. Reducing the expenses of food, by:
      (1) Budgeting funds allotted
      (2) Purchasing at sales and within seasons
      (3) Shopping to find lowest prices
      (4) Making home gardens
      (5) Raising chickens, rabbits
      (6) Canning fruits and vegetables
      (7) Using left-over foods
      (8) Baking bread, cakes, and pie at home
      (9) Planning weekly menus.
   b. Reducing the expenses of clothing, by:
      (1) Sewing clothing at home
      (2) Caring for clothing, pressing, cleaning, mending
      (3) Using quality standards in making purchases
      (4) Planning costumes for occasions.
   c. Reducing the expenses of shelter, by:
      (1) Moving to a lower rent area
      (2) Buying a home in some cases or renting in others.
   d. Reducing operating expenses, by:
      (1) Being economical about the use of lights, radio, water, and electrical instruments and utensils
      (2) Using driftwood for the fireplace
      (3) Making no unnecessary long-distance telephone calls
      (4) Making small repairs, painting, caring for the lawn yourself.
   e. Reducing miscellaneous expenses, by:
      (1) Using club offers and the public library instead of buying for one reading
      (2) Providing home amusements, games, recreation
      (3) Earning spending money
      (4) Carrying group insurance of essential kinds—health, accident, Blue Cross.
   f. General measures to reduce expenses
      (1) Joining cooperatives
      (2) Joining 4-H clubs and similar organizations producing things
      (3) Using dependable advice when making purchases (Consumer Research Bulletins)
      (4) Using a family and a personal budget
      (5) Keeping accurate accounts of expenses and receipts
      (6) Buying for cash versus instalment buying
      (7) Taking advantage of discounts for prompt payment of bills

(8) Eliminating charge accounts
(9) Safeguarding cash on hand.

g. The mathematics involved in the study of expense reduction
   (1) Computations with whole numbers, fractions, and decimals as needed
   (2) Numerous applications of percentage, including discount, interest, profit
   (3) Construction of graphs, tables, charts, diagrams
   (4) Making applications of many forms of measurement
   (5) Mathematical recreations, games, puzzles.[6]

The types of activities engaged in by the students are grouped under the headings given below:

1. Problem-solving activities
   a. Formulated problems for study
   b. Considered ways of securing information needed
   c. Gathered information, exhibits, and other materials related to problems selected by groups for study
   d. Organized information and materials for reporting to the class
   e. Presented and evaluated reports, materials, exhibits
   f. Formulated conclusions and generalizations
   g. Participated in several debates, panel discussions, round tables, town meetings
2. Constructions and concrete experiences
   a. Collected clippings, advertisements, bills, statements, account forms
   b. Collected recipes for nutritious foods
   c. Made electric- and gas-meter forms of cardboard
   d. Kept personal and family accounts
   e. Prepared exhibits of materials to make reports concrete
   f. Examined and tested fabrics in home-economics class
   g. Earned money in a variety of ways
   h. Collected book covers for the "What to Read" poster
   i. Started gardens, raised chickens at home
3. Excursions
   a. Visited various places of business and sales
   b. Visited a poultry-raising establishment, a canning factory, and orchards
   c. Visited community-foods kitchen to see the large-scale food preparation
   d. Visited employment bureau to study placement procedures
   e. Visited the Blue Cross office and insurance offices

[6] *Ibid.*, pp. 98–99.

4. Appreciation and creative activities
   a. Read many vital, informative books and stories
   b. Made a variety of booklets of materials, pictures, and clippings
   c. Viewed several films and sets of slides
   d. Listened to advertising on radio broadcasts and evaluated it
   e. Learned some interesting games for home recreation
   f. Considered the question of what recreation the community should provide
   g. Evaluated costumes appropriate for various occasions, their cost and merit
   h. Made scale drawings of prospective homes, patterns of costumes, and plans of gardens
   i. Prepared pictures, paintings, and draperies to beautify the home
5. Practice activities
   a. Used computations many times in the course of the unit and kept a progress-test record
   b. Had many contacts with social applications of arithmetic and other branches of mathematics
   c. Had numerous contacts with business practices and came to understand them
   d. Constructed many different visual materials for presenting information, including pamphlets, bulletins and charts
   e. Made extensive use of reading skills in locating information needed
   f. Showed relations between the work in arithmetic and that in home economics, social studies, and other areas
   g. Engaged in a meaningful, group-cooperative enterprise
   h. Had much functional practice in use of language skills, oral and written.[7]

Short descriptions of three other units follow.

I. The Cost of Owning an Automobile Today[8]

1. Overview

The purpose of this unit is to give the pupils a better understanding of the cost of the family automobile, its more frequent operating costs, and some of the other expenses such as licenses and insurance policies which are incidental today to the owning of an automobile.

[7] *Ibid.*, pp 100–101.

[8] Adapted from a resource unit developed by Robert D. Anhorn, graduate student at the University of Minnesota

Through a variety of projects and activities, the pupils will learn that the automobile is not only a large investment for the average family but requires a great deal of care and expense. For most pupils of this age, the unit provides an introduction to the value and upkeep of an automobile, which they will be driving in a few years.

This study offers many opportunities in the use of the basic computational skills. The integration of the mathematical and social phases of arithmetic is quite evident in most of the activities of the unit. In addition, there is much practice in language, reading, research, and problem solving. The students can participate in the planning of the activities to be carried on in connection with this unit.

2. Initiating the Unit

With the introduction of many new model cars late in the fall of the year, an experience unit relating to the automobile would be quite appropriate and easily introduced to the class.

An understanding of the cost and care necessary to a car would not be so evident among the pupils.

Some pupils may have raised questions concerning the estimated costs of accidents as reported in the newspaper. Questions about insurance may also be raised in this connection.

Another pupil may want to know the meaning of winterizing a car and the cost of doing so.

The teacher could use reports in newspapers about automobile shows or displays, accidents due to careless driving by young people, and similar materials to introduce the subject. The subject of the unit also may grow out of a discussion of family budgets.

3. Contents of Unit

a. The necessity of the automobile in modern community life
   (1) Family life
   (2) Recreation and travel
   (3) Business and industry
b. The original costs of an automobile
   (1) The cost of a new car
   (2) The cost of a used car and factors affecting the cost
c. Factors to be considered in buying a new or used car
   (1) Financial status of buyer
   (2) Use to be made of vehicle
   (3) Length of service expected
   (4) The reliability of the dealer
   (5) The time to buy
   (6) Value received

d. Methods of financing
   (1) Paying cash (with trade-in of old car)
   (2) Loans by banks and financing companies; private loans
   (3) Repayment plans
   (4) Special arrangements
e. Factors affecting the price of a car
   (1) Quality, appearance, efficiency of operation, size, model
   (2) Federal (and state) tax included in purchase price
   (3) Cost of transportation of car to local dealer—how determined
   (4) "Extra" costs of special equipment—bumper guards, heater, radio, etc
f. Regular operating costs
   (1) Motor oil
      (a) Need of motor oil
      (b) Kinds, quality, costs
      (c) Buying in bulk vs in containers
      (d) Amount in a change of oil, frequency
      (e) Oil filters
      (f) Seasonal changes necessary
   (2) Gasoline
      (a) Kinds of gasoline required in different makes of cars
      (b) Prices of different types of gasoline charged by gasoline stations and causes of the variations
      (c) State, federal, and local taxes
      (d) Uses of funds raised by taxation
      (e) Mileage
      (f) Thrifty driving
      (g) The approximate annual cost of gasoline used in driving an average car 8000 miles
   (3) Protecting the car
      (a) Lubrication
      (b) Need and cost of anti-freeze—permanent and temporary
      (c) Washing
      (d) Garage and storage
   (4) Depreciation in value
g. Other incidental costs in operating a car
   (1) Storage battery
      (a) Costs
      (b) Length of life
      (c) Case of battery
   (2) Tires and tubes
      (a) Costs
      (b) Length of life—distance; guarantee

(c) Rotation—frequency and need
(d) Recent developments to increase safety
(3) Spark plugs
(4) Costs of repairs of various kinds and replacement of parts
(5) Driver's license
(a) Cost
(b) Method of receiving license
(c) Nature of tests, if any
(d) Renewal of licenses
(6) Automobile license
(a) Cost of license for cars of different types
(b) Basis of the cost—weight, model, year, value
(c) Uses of funds raised
(7) Painting and incidental repairs
(8) Parking; metered charges
h. Automobile insurance—kinds, necessity, costs
(1) Fire, theft, storm
(2) Personal liability and property damage
(3) Collision
(4) Medical care
(5) Comprehensive policies
(6) Compulsory insurance in certain states
i. Services of automobile clubs and costs of membership

4. Activities during the Unit

a. Discussing personal experiences suggested by the contents of the above outline of subject matter
b. Gathering comparative data about the costs of new and used cars
c. Securing actual data about costs of various ways of financing the purchase of a new car
d. Studying costs of state car licenses
e. Preparing a bulletin board of pictures, clippings, and the like showing necessity of automobile insurance
f. Preparing a list showing the total cost of various kinds of automobile insurance judged to be necessary
g. Showing relation of speed of driving to economy of fuel
h. Visiting a museum or looking at a display of the historical development of the automobile
i. Keeping a record of the actual expenses of operating a family car, including depreciation of value
j. Discussing ways of reducing the expenses of operating and maintaining a car
k. Discussing the uses of the funds raised by license fees, including the cost of building and maintaining highways

l. Discussing new ways of paying for turnpikes, including toll charges
m. Discussing the steps involved in securing a driver's license and the tests to be taken, if any
n. Debating the pros and cons of compulsory automobile insurance
o. Developing and applying various formulas. (See Chapter 12.)

II PROPERTY TAXATION

1. Objectives
   a. To teach the meaning and social significance of taxation
   b. To develop an understanding of the services provided by the tax dollar
   c. To teach the meaning of taxation, how the rate is determined, and the method of determining the taxes on a piece of property
   d. To increase the student's knowledge of local government
   e. To help the students to sense what can be done to increase the efficiency with which money raised by taxation is spent
2. Initiating the Unit

The approach to the study of taxation in one school was through a study of the procedures used to determine the dues students pay to the student organization. The governing body of the organization consists of the elected officers, and representatives of each home room, and club or activity sponsored by the school Near the close of the school year the student organization proposes a budget for the following year. Each club submits a list of anticipated expenditure for the coming year A budget committee of students and a faculty member studies these requests At a special "budget hearing" requests are presented and discussed and then adjusted as may seem necessary Then the recommended budget is presented to the student body as a whole. Each student receives a copy of the detailed budget. After a discussion of the items in it and of changes that may be desired by the student body to adjust the dues, the students vote to accept or reject the budget. The individual dues are found by dividing the total of the budget by the number of students enrolled

The yearly dues are collected by a student selected in each home room at the time of registration at the opening of school. A discount is allowed for early payment. Students who pay their dues are admitted to the activities sponsored by the student organization and have other benefits of regular membership, including admission to athletic events, copies of school paper, and so on.

3. Contents of the Unit

A study of local government in relation to property taxation—town, city, or country (leading to a study of other kinds of taxation)

a. The governing body—council or commissioners
   (1) How selected
   (2) Duties of each member
   (3) Duties of heads of branches of local government, such as public safety and welfare, public works, parks and public buildings, finance, and public relations
   (4) Independent boards, such as school board, library board, etc.

b. How the local budget is prepared
   (1) Preparation of departmental budget, showing the various services rendered
   (2) Combining departmental budgets
   (3) Public hearings or town meetings at which taxpayers discuss the proposed budget, possible reductions, and ways of increasing the efficiency of public expenditures
   (4) Revision of budget as may be advisable

c. How the local tax rate is determined
   (1) Method based on assessed valuation
   (2) Finding the rate by dividing budget total by assessed valuation
   (3) Ways of expressing rates

d. How taxes for property owners are determined

e. How taxes are collected
   (1) Tax statements
   (2) Penalties for delinquents
   (3) Discounts for advance payments

f. How the tax dollar is split
   (1) Police and fire protection
   (2) Streets, sidewalks, water
   (3) Education and libraries
   (4) Courts
   (5) Health
   (6) Parks and recreation
   (7) Garbage removal
   (8) Others

g. General discussion of other forms of local, state, and federal taxation may follow the study of property taxes, along similar lines
   (1) Sales tax
   (2) Gasoline tax, luxury, amusement, transportation

(3) Income tax
(4) Customs, duties, tariffs
(5) Licenses and fees
(6) Hidden taxes

4. Student Activities Connected with Unit

a. The development, evaluation, and adoption of the school budget by the students
b. Discussion of services provided locally by the tax dollar as shown in tax reports, newspaper articles, etc.
c. Making tables, graphs, and diagrams showing distribution of tax dollar
d. Excursion to City Hall and other public buildings where various branches of the government have their headquarters
e. Comparing local tax rates with rates in other communities and discussing possible reasons for the differences
f. Comparing local cost of education with costs of other governmental activities and considering local needs
g. Making cuts, posters, displays for PTA meetings
h. Considering ways in which to reduce the expenses of the school and of maintaining public property
i. Viewing the film, "Property Taxation" (Encyclopaedia Britannica), and discussing its contents
j. Studying contents of the regular mathematics textbook and solving problems about taxation it contains which have been made meaningful by the work done in the unit
k. Formulating a list of basic concepts and words essential to the understanding of the topic
l. Summarizing the relationships between the operation of student government and municipal government

III. Unit on Insurance

1. Objectives

a. To give the students an understanding of the purpose and usefulness of insurance
b. To develop attitude of foresight and preparedness in dealing with problems of everyday living
c. To demonstrate to students some of the advantages of cooperation among people
d. To familiarize students with the meaning of many new technical terms
e. To teach the students about the simpler elements of mathematics on which insurance is based. (See pages 260 to 266.)

2. Initiating the Unit

This unit should begin with an exploration of what the students already know about insurance and its role in everyday life. Some local happening, such as a fire, an accident, or damages caused by a storm can provide the initial approach to the study. Questions may be raised about insurance policies that families, individuals, and businesses have about which the students know. The teacher can explore the students' background to discover misconceptions they may have about the nature of insurance, the aspects of the topic in which they may have a special interest, and activities they might like to carry on in connection with their study of the subject.

3. Contents of the Unit

The content of a unit on insurance can include such topics as the following, adjusted as may be necessary to the needs, interests and level of development of the students and as local circumstances may warrant:

a. Property insurance
   (1) Real property, insured against misfortunes that may befall a family or organization
   (2) Personal property
   (3) Crop insurance
   (4) Automobile insurance, including liability, property damage, fire-theft, collision, medical, basis of claims
b. Personal insurance
   (1) Various types of life insurance
   (2) Health, accident, and hospitalization
   (3) Income replacement
   (4) Social security
   (5) Liability
c. The mathematical basis on which premiums are established. (See the discussion of the mathematical basis of insurance given on pages 260 to 266.)

4. Pupil Activities

a. Listing situations showing the need of insurance in specific instances based on student experiences
b. Making a list of the insurance needs of a family; of a student
c. Studying typical insurance policies of various kinds
d. Discussing the objectives, costs, and benefits of the various kinds of insurance
e. Listening to a discussion by an insurance expert
f. Viewing films about insurance, such as "For Some Must Watch," Coronet Films, 1949
g. Studying filmstrips, such as the series issued by the Institute of Life Insurance, 1951.

Sources of Other Units

Many school systems have conducted workshops in which teachers have prepared source units that have been published and can be secured for a small sum. There also are available excellent monographs which contain detailed descriptions of instructional units. For example, the monograph, *Illustrative Learning Experiences*,[9] contains the following units:

1. Donovan Johnson, Our Number System A Unit on Algebra, Pages 62–69.
2. Theo. E. Kellogg, Mapping as Applied Geometry A Unit in Geometry, Pages 70–79.
3. Ramon P. Heimerl, Buying Insurance A Unit in Basic Business, Pages 52–61.

[9] The Modern School Practices Series, No 2 Minneapolis University of Minnesota Press, 1953.

Questions, Problems, and Topics for Discussion

1. Why is the traditional daily page-by-page assignment of so little value? Why is it used so widely today?

2. Examine mathematics textbooks to discover the extent to which the contents are organized as units. Illustrate how.

3 Can you mention typical enterprises and organized school experiences in which you have participated that give the students experience in the types of organized institutions in life outside the school in which number plays an important role? Discuss ways of developing in some school any one of the six enterprises listed on page 272. Can you describe others? What is the value of such enterprises? What is "Junior Achievement?" What is the value of 4H Club activities from the point of view of mathematics?

4. Why may it not be desirable to attempt to categorize units as to type? Why may it be better to consider the characteristics of the most valuable kinds of learning experiences and include any and all types in a unit whenever appropriate?

5. How can the teacher proceed in the selection of suitable instructional units? What criteria should be considered in judging the value of a particular unit? Apply them to one or more of the units described in this chapter. Observe a lesson and evaluate both content and method

6 Illustrate the meaning of each of the twelve themes listed on pages 276 to 277. How can they be incorporated in the study of some topic, such as taxation, banking, or the cost of operating an automobile? Examine these units for illustrations Can the twelve themes be included in any single unit? Try to develop some unit in which all of the themes are included

7. Outline the contents of a typical subject-matter unit of the more traditional type.

8. Observe a lesson in which a new step in some operation is taught. Apply the sequence of procedures given on pages 279 to 280 to the lesson.

9. Develop a lesson plan for teaching some new step in percentage or decimals. (Refer to related chapters in this book.)

10. Develop an outline of a projected experience unit following the steps discussed in this chapter. Follow the generalized outline given on page 287, or a similar plan used locally. Include a variety of the kinds of activities listed on pages 285 to 286.

11. What is an "overview"? Develop standards for evaluating an overview. Apply them to the overviews included in this chapter.

12. How can the students participate in the planning of the activities engaged in during a unit?

13. Examine courses of study to determine helps given teachers in unit teaching. Read descriptions of units that have been published by school systems.

14. Look up available "source units." They often contain a wealth of information of value to teachers.

Suggested Readings

Alberty, H. *Reorganizing the High School Curriculum* (Revised), Chapter 9 and Part IV. New York: The Macmillan Company, 1953.

Brueckner, L. J. and Grossnickle, F. E. *Making Arithmetic Meaningful*, Chapters 4 and 5. Philadelphia: The John C. Winston Co., 1953.

Burton, W. H. *The Guidance of Learning Activities* (Revised), Part III. New York: Appleton-Century-Crofts Co., 1952.

Giles, H. H. *Teacher-Pupil Planning*. New York: Harper and Brothers, 1941.

Morrison, H. C. *The Practice of Teaching in the Secondary School* (Revised). Chicago: University of Chicago Press, 1931. Classic reference on subject-matter units.

Schorling, R. *Student Teaching* (Revised). New York: McGraw-Hill, 1949.

Strickland, Ruth G. *How to Build a Unit of Work*. Bulletin No. 5, U. S. Office of Education. Washington, D. C.: Government Printing Office, 1946.

Thut, I. N. and Gerberich, J. A. *Foundations of Method for Secondary Schools*. Chapters 9–14. New York: McGraw-Hill, 1949.

Umstattd, J. G. *Secondary School Teaching* (Revised). Boston: Ginn and Co., 1944.

Chapter 9

Improving Problem Solving and Quantitative Thinking

This chapter deals with the following topics pertaining to problem solving:

a. What is a mathematical problem?
b. Levels of problems
c. Auxiliary skills in problem solving
d. Teaching problem solving at Levels I and II
e. Solving problems classified as Level III
f. Solving problems classified as Level IV.

a. What is a Mathematical Problem?

The Difference between Problems and Examples

How to solve mathematical problems and how to teach students to solve them are problems every teacher faces. What is a mathematical problem? A student encounters a mathematical problem when he confronts a quantitative situation which he cannot answer in a habitual manner. It follows, then, that what may be a problem for one student is not a problem for another student. What may have been a problem for a student yesterday may not be a problem for him today. A student's background and experience with quantitative situations determines whether a situation presents a problem to him. Furthermore, the student's

background of experience and his mental capacity determine whether he has the knowledge and *ability* needed to solve the problem.

Many teachers of arithmetic think of verbal problems in a textbook as "problems" and an indicated operation as an "example." Both forms may be classified as either a problem or an example, depending upon the background and ability of the student. For a student in the junior high school, neither of the following should constitute a problem:

a. How much are 7 eights? or, $7 \times 8 = ?$

b. What is the cost of 7 yards of ribbon at 8¢ a yard?

In each of these quantitative situations, the student may be able to give an almost immediate response. In the verbal problem in b, the student has to identify the process to use, but in a the process is indicated. However, the student may have had so many experiences with problems of the kind in b that he at once identifies the process as multiplication.

The student who can give an immediate response to the question, "How much are 7 eights?", may encounter a problem if the question is changed as follows: Give at least three ways to prove that 7 eights are 56. If a habitual response is not possible, the question becomes a problem requiring quantitative thinking.

Backgrounds of Students Important in Problem Solving

Let us consider the backgrounds of experience of two junior high school students in studying the multiplication facts for review purposes. One student may have memorized these facts as tables and the other student may have learned the facts by discovery of meanings and relationships among numbers. The student who learned the facts mechanically by rote as isolated elements may not be resourceful enough to give three ways to prove that 7 eights are equal to 56. He learned by rote the fact, $7 \times 8 = 56$, as an element in a multiplication table. Now he is faced with a problem to which he cannot give a habitual answer. His success in solving the problem depends upon the degree of understanding he has of multiplication facts.

The second student may have learned the basic facts in multiplication by discovery of relationship among them. He learned how to find the answer to a number grouping in this process in many different ways. This student may have learned that 8 eights are 64 and that 7 eights would be 8 less than 64. Similarly, he may have learned how to group eights and from that knowledge he should be able to find the answer to 7×8. This student may have discovered that the only two multiplication products of the basic facts in the 50's are 54 and 56. He knows that the answer to the grouping, 6×9, is 54, hence the answer to the grouping, 7×8, must be 56. This student thus became resourceful in finding the answer to number groupings; hence it is safe to predict that he could offer several different ways to prove that 7 eights are 56. The differences in the backgrounds of the two students in their work in multiplication would differentiate their behavior in dealing with the problem.

Both students faced an arithmetic problem. Both students identified the goal, but one student was able to reach that goal because of his enriched understanding of multiplication Both students could give the answer to the number grouping provided no quantitative thinking was required. The same situation applied to the solution of the verbal problem involving the fact. The element which presented the problem for the students was the attainment of the goal which was not immediately evident to them. The form in which the question was given was of minor importance to the student. The solution of the problem was not appreciably affected either by the indicated notation of the example or by the verbal statement. The type of response given in a quantitative situation indicates whether or not the question raised constitutes a mathematical problem for the student.

Essential Elements of a Problematic Situation

Thorndike has stated that there are three elements common to all problems. They are:

1. The individual is orientated towards a particular objective and motivated to reach it. He has an end in view.
2. Progress towards the objective is blocked.

3. Available, habitual response patterns are not adequate to permit the individual to surmount the obstacle and proceed toward his objective.[1]

According to the third characteristic of all problems, the individual has no habitual response pattern available for solving the situation. The degree to which the student is able to offer a variety of possible solutions affects the chance of finding a satisfactory solution. Not only should the student be able to suggest a variety of possible solutions of the problem but also he should be able to test these hypotheses for feasibility. The testing of hypotheses may be given as a fourth element in problem solving. Thus, the elements of a problematic situation which the student identifies are:

1. A desired goal is to be attained
2. There is a blocking of the path for attaining the goal
3. Habitual responses are not sufficient to attain the goal
4. Various hypotheses or solutions are offered and tested.

It is evident that the student who is able to offer a number of possible hypotheses or solutions has a splendid chance of success in solving a problem. On the other hand, the student whose background is so limited that he must look for a word or some other cue in the statement of a problem which will enable him to decide what procedure to follow in solving it has little likelihood of success in solving a problem.

Discovering a Familiar Situation in Verbal Problems

A student who discovers a relationship between a new problem situation and one that is familiar to him is likely to be able to solve the new problem. If he is unable to discover this familiar pattern or model in the new situation, he will find it difficult to solve the problem. The student may feel certain that the new problem is different from any that he has ever solved. With a mental attitude of this kind he may not be able to solve the

[1] Thorndike, Robert L. "How Children Learn the Principles and Techniques of Problem Solving," *Learning and Instruction*, p 193. Forty-ninth Yearbook of the National Society for the Study of Education, Part I. Chicago· University of Chicago Press, 1950.

unfamiliar problem. The ability to show how situations differ does not lead to success in problem solving. He must continue his search for a familiar pattern or model concept in the new situation. In other words, it is necessary for the student to see the similarities as well as the differences in a situation.

A group of college seniors who were preparing to teach in the elementary school were given the following problems:

1. Two students go to the "snack bar" and each has the same luncheon. One student buys an extra cup of coffee for 10¢. If one check for the total bill is $1.10, what is the cost of the luncheon?

2. The sum of two numbers is 38. If one of the numbers is 4 more than the other number, what is each number?

3. An airplane can travel 180 m p.h. with a tail wind and 130 m.p.h. with a head wind. What is the speed of the wind?

4. A boat can go upstream 10 m.p.h. and downstream 15 m.p.h. What is the speed of the current?

An analysis of these problems shows that the mathematical principles involved are similar. Problems 1 and 2 are almost identical and problems 3 and 4, also, are identical. It is difficult to find a college senior who cannot give an immediate answer to the first problem. Many students find the second problem much more difficult than the first problem. Some students are unable to solve the second problem without using an algebraic solution. Why is there such a difference in the difficulty of the two problems even though both are representative of the same principle? The reason is due to the experiences the students have had with these two quantitative situations. Two students in the class often bought sandwiches and coffee under the circumstances stated and each shared his part of the cost. These students did not see that the principles involved in the problem about the sum of two numbers is the same as the principle involved in the problem about the cost of the lunches. In the first problem the cup of coffee is the quantity which makes the costs of the two lunches unequal, while in the second problem the number 4 is the quantity. On the other hand, if the students had discovered that the two problems are almost identical in principle, they would have had no difficulty in finding the two numbers.

In problems 3 and 4, the quantities must be found, while in problems 1 and 2 the quantities are given. The writers' experience with seniors in college shows that the students recognize these problems as involving an algebraic solution which seldom is remembered or understood. When the students discover how these problems are related to the familiar problem of the luncheon and the extra cup of coffee, the solutions are easy. It is apparent that the secret of problem solving consists in having a meaningful experience which is related to the problem to be solved. As soon as a student discovered the familiar pattern or model which is common to all these problems, he was able to solve all of them.

Classification of Problem Situations

Many teachers assume that problem solving consists solely in solving verbal problems found in a mathematics textbook. This interpretation represents a narrow conception of problem solving. Such a restricted view is inadequate as the basis of a discussion of problem-solving ability. There are teachers who hold to the view that a problem must have social significance in order to meet the standard of a verbal problem. Then, if a verbal statement asks the student to identify a mathematical relationship which has no social significance, the verbal statement is regarded as an example and not as a problem. The difference between the two forms of verbal statements may be illustrated as follows:

1. How many sixes are there in 42?
2. How many 6-inch pieces of ribbon can be cut from a piece of ribbon 42 inches long?

If a verbal problem must have social significance, question 1 is an example, but question 2 is a problem. Until a student reaches the level of ability in dealing with quantities when he can give an immediate response to either question, each of the statements constitutes a verbal problem. A student deals successfully with a problem when he identifies the common relationships between two or more quantities or shows how one quantity depends upon some other quantity. The factor of social significance is not a necessary condition for a verbal problem.

b. Levels of Problems

A Basis of Classifying Problems

Four levels of teaching and solving mathematical problems can be identified. *Problems illustrating the first three of these levels may be found in mathematical textbooks, but problems illustrating the fourth level are found in real life situations.* The four levels are as follows:

LEVEL I. Verbal problems at this level appear in a miscellaneous order. There is no relationship between consecutive problems in a problem-solving exercise. The verbal problems found in a typical standard test in problem solving exemplify this level. Most students must take tests, civil service examinations, or other forms of evaluation to establish rankings in number competence. A limited number of miscellaneous exercises dealing with problems of this type can be justified. These problems are essentially "disguised drill."

LEVEL II. At the second level, verbal problems are grouped according to topics or processes. For instance, after a process is taught, such as finding a per cent of a number, verbal problems are given which involve the principle of finding a per cent of a number. Often they are grouped around some topic, such as the "Uses of Per Cent." It is evident that problems of this kind do not offer or provide much opportunity for growth in ability to deal with quantities. To a large extent, problems of this kind are examples couched in verbal statements.

LEVEL III. At the third level, problems deal with units having social significance. The student may be asked to interpret a graph, evaluate data in a table, or solve problems growing out of information collected about a given topic. At the junior high school level, problems about topics such as the following illustrate this type:

1. A century of progress in transportation
2. What the farmer receives for eggs or for milk
3. Land uses in the United States
4. Saving soil by crop rotation
5. How our eating habits have changed

6. The shrinking value of the dollar
7. How machines have increased time for leisure
8. Shift of population in our country
9. How you help support the government
10. Bins for storing corn.

The problems found in any unit of this type are related. Each problem helps to enrich the student's information about a given topic and to direct his thinking so as to discover relationships among quantities. Problems of this kind are not disguised drill, but instead they help provide the student with vicarious experiences in dealing with a topic which is thus made both mathematically meaningful and socially significant to him.

In dealing with a topic such as measurement, the student should use materials which will enable him to discover the rule to express the area or volume of a geometric figure. Then he should be able to derive a formula from that rule. (See Chapters 11 and 12.) The formulation of generalizations and the discovery of relationships among numbers represent a high level of problem solving.

Level IV. At this level the student deals with real lifelike problems. He devises methods of solutions and collects data which may be used to help solve a problem having social significance to him or his class. The class makes use of community resources to solve problems at this level. Textbooks in mathematics can provide very few problematic situations classified at this level. Typical problems of this kind are as follows:

1. What rate of interest is paid on deposits and when is it paid by banks in your community?

2. What kinds of banks are located in your community? How do these banks differ?

3. How is the tax dollar spent in your community or your county?

4. What is the cost of education per pupil in your community compared with the cost in surrounding communities?

5. How is the money raised to maintain our schools?

6. What's the best way to make a trip from Chicago to New York?

Problems classified at the fourth level cannot be provided in textbooks, although they may be suggested by the topics included. The student has to collect data bearing on each topic, delete facts which are irrelevant, and interpret the results. Problem solving at this level typifies problem solving in real-life situations. Most adults have to make decisions about topics concerning which it is necessary to collect pertinent facts or information in order to form mature judgments.

There is a place in the classroom for problems which may be classified under each of the four levels. Most of the verbal problems in a textbook in arithmetic should be classified under the third level. The worth of such a textbook varies directly as the percentage of verbal problems that grow out of units which have social significance. The teacher should know how to deal with problems at each of the four levels mentioned. The following pages give some suggestions for teaching problems that may be classified under one of these levels. All of these types refer to problems which may be solved in group activities.

Highest Level of Problem Solving

The highest type of learning involving problem solving consists in having the student work independently of the class. He achieves mastery of some topic not in the regularly assigned work of the class. For example, the student may learn how to transform a magic square or how to use a slide rule. The class may have had some work with magic squares, but no work about how to transform them. The teacher may suggest the title of a book which deals with this topic. The student who explores the topic and masters the process of transforming magic squares by himself exhibits the highest possible level of problem solving.

The class may not have discussed the use of a slide rule, but a student may have one and a manual of instructions dealing with the operation of such a rule. If he is able to interpret and follow the instructions explaining how to use a slide rule, he has developed good study habits and learned to read with understanding. A student who learns to work independently and to

educate himself demonstrates a very high and desirable type of quantitative thinking which will serve him well in later years throughout his life.

Problem Solving Is Not a Skill

It is not possible to teach problem solving as a skill. If a student learns mathematics meaningfully, he must solve problems as he deals with quantitative aspects of real or problematic social situations. All genuine learning situations in mathematics involve problems. When a student uses objective and visual materials to solve problems that otherwise he could not solve by symbols, he is doing a valuable kind of quantitative thinking. The student who does not discover relationships among numbers by using objective materials merely manipulates the objects. It is for this reason that the teacher was cautioned in Chapter 4 about the use of objective materials in the classroom. If a student is able to operate effectively with symbols, he need not use objective or visual materials. Problem solving is a continuous growth process in any effective program which emphasizes meaning and understanding.

Many teachers view problem solving as the ability to solve verbal problems as found in a textbook in mathematics. These teachers look upon problem solving as a special skill which is different from the ability to master computational procedures of mathematics. The authors believe that problem solving is not a specialized skill the student should develop or master. The old familiar cliché, "My students are good in computation but they cannot solve problems," is misleading. As a matter of fact the student is good in neither to the exclusion of the other. If he is "good" in computation, but poor in problem solving, he does not understand mathematics. His knowledge of number is typical of the student who says, "If you tell me the formula or the equation to use, I can solve the problem." In a mechanical manner this student can perform computations, substitute values in a formula, or solve an equation, but he does not understand what the equation represents or what the basic principles are which govern its solution. There is no real dichotomy between

computation and problem solving. The two processes are interwoven. The teacher who emphasizes the one to the exclusion of the other is not providing a favorable learning situation for the student.

Accuracy in Computation Needed in Problem Solving

It is possible for a student to make a low score on a standardized test in problem solving due to inaccuracy in computation. He may use the correct procedure in solving a problem but get an incorrect answer due to a computational error. In most tests of problem solving, the measure of problem-solving ability consists in finding the number of correct answers to problems. Problem-solving ability should be measured by the ability of the student to use the correct method of solution. However, the student must regard accuracy in computation with the basic processes as essential. When the student makes errors in computation, the teacher should diagnose the difficulty and correct it by the techniques discussed in the chapters dealing with the basic processes with integers, common and decimal fractions, and per cents. Special help in diagnosis is given in Chapter 13.

Even though it sometimes is difficult to teach problem solving, teachers of mathematics must develop this ability in their students. In order to help the teacher, we shall consider how to teach problems classified under the four levels described.

c. Auxiliary Skills in Problem Solving

Developing the Arithmetic Vocabulary of the Student

There are certain factors which affect the student's ability to solve problems classified under levels I and II. The first of these factors is the extent of the student's *vocabulary*. *Mathematics* is a science. Every science includes a vocabulary which is distinctive. The use of an important but unfamiliar word greatly increases the difficulty of solving a verbal problem. The answer found to a problem containing an unfamiliar technical term cannot be checked by the student to see if the answer is sensible.

A group of 19 college seniors solved the following problem:

A fowl loses $\frac{1}{3}$ of its weight in dressing. How many pounds of fowl are needed to produce 10 pounds of dressed meat?

Every student in the group gave $6\frac{2}{3}$ pounds as the answer. The instructor anticipated that some of the students would find $\frac{1}{3}$ of 10 pounds, add $3\frac{1}{3}$ pounds to 10 pounds, to make a total of $13\frac{1}{3}$ pounds. Instead, all of the students subtracted $3\frac{1}{3}$ from 10, and gave $6\frac{2}{3}$ pounds as the answer. Then he asked the students to explain why they subtracted $3\frac{1}{3}$ pounds from 10 pounds. A typical answer was as follows: "The problem calls for *dressed* fowl. When you dress a fowl, it weighs more than when there is no *dressing* in it. Therefore, you would need less undressed fowl than dressed fowl. So we subtract $3\frac{1}{3}$ from 10." All of these students lived in a city. The term "dressed fowl" did not convey the meaning to them that it would convey to adults who have had experience in preparing poultry for retail sale. After the term was defined to the students, they said, "You mean the fowl was cleaned or drawn." When the meaning of the term was clarified, the students gave the correct answer, namely, 15 pounds. It is significant to note that there was a very definite cue in the problem which stated that the fowl *lost* one third of its weight in dressing. This mathematical term should have indicated to the students that the weight of the live fowl would be greater than the weight of the dressed fowl. The students did not sense the evident cue in the problem. Their experience was with adding dressing to a fowl, which increases its weight; hence they subtracted rather than added to find the weight of the live fowl.

Horn reported that mathematical terms used in the social studies are often not understood by students. As an illustration, students gave the following interpretations of the concept "ten square miles." "Ten square miles meant to various pupils *about the size of Chicago, about the size of the State of Iowa, about the size of Washington Park, as large as ten acres,* and *here to Key West in a straight line.*" [2]

[2] Quoted by Horn, E. "Arithmetic in the Elementary-School Curriculum," *The Teaching of Arithmetic*, p. 12. Fiftieth Yearbook of the National Society for the Study of Education, Part II. Chicago. University of Chicago Press, 1951.

If a student has such an erroneous concept of the meaning of the term ten square miles, it is evident that he cannot use that concept intelligently in a quantitative situation. The teacher should be certain that the students understand the concepts used in a problem.

Johnson[3] suggested a list of activities that may be used with students to enrich their understanding of words used in mathematics. Some of these activities included the following:

1. Using the dictionary to find the meanings of words
2. Using given words in sentences
3. Matching words with definitions, objects, pictures
4. Grouping or classifying mixed lists of words under proper headings
5. Naming the unit of measure or the instrument used in measuring various items or aspects of things
6. Giving words having similar meaning
7. Naming geometric figures or parts of drawings, or drawing representations of expressions
8. Performing some action to show meaning
9. Restating expressions in other words
10. Giving opposites or synonyms of words.

When a student completes a unit of work, he should be given a vocabulary test to see if he understands the concepts used in the unit. The test on page 314 is typical of the kind to use for this purpose. This test is found near the end of a chapter dealing with certain plane figures and their properties.

The list of concepts given there includes the technical terms which the student meets in that particular textbook in the chapter dealing with plane figures. If the student demonstrates that he understands the meaning of a concept, the use of that concept in a problem is not likely to create a block because of his unfamiliarity with the expression.

The directions provide for three different levels of ability in dealing with a concept. The superior student should be able to formulate his own definition of a term. On the other hand, the slow learner who might not give a definition of the term could use it in a sentence.

[3] Johnson, H. C. "The Effect of Instruction in Mathematical Vocabulary upon Problem Solving in Arithmetic," *Journal of Educational Research* 38.97–111.

The Vocabulary of Arithmetic[4]

Define, illustrate, or use in a sentence each of the terms given below. The number in parentheses shows where the term is used.

Acute angle	(141)	Obtuse angle	(141)
Altitude	(138)	Parallelogram	(144)
Area	(129)	Protractor	(142)
Central angle	(155)	Radius	(149)
Circumference	(149)	Right angle	(131)
Diameter	(149)	Scalene triangle	(140)
Equilateral triangle	(140)	Straight angle	(141)
Formula	(127)	Trapezoid	(146)
Isosceles triangle	(140)	Vertex	(143)

Such concepts as "speed" or "average speed" are not understood by many students at the junior high school level. A supervisor observed seventh grade students solve the following problem: What must be the average speed of a car which travels 105 miles in 3 hours? Several of the students gave 35 miles as the answer. When the papers were scored, one student asked if the answer was 35 miles or 35 m.p.h. The teacher said the answer was 35 m.p.h. There was no attempt made to have the student understand the difference between distance and speed or average speed. A discussion to clarify these concepts would have been more profitable than solving a page of miscellaneous problems as an exercise in problem solving.

Improving the Reading Ability of the Student

The second factor affecting a student's success in solving problems depends upon the level of his reading ability. A student's reading ability is very closely related to the extent of his vocabulary. The ability to read a verbal problem is quite different from the ability to read a short story. In recreational

[4] *Winston Arithmetic: The New Knowing about Numbers*, p. 160. Philadelphia: The John C. Winston Co., 1956.

reading the student may be interested in the plot or theme with which the story deals. He may skim the page or at least he should read rapidly. In reading a verbal problem the student should learn to read carefully and slowly so as to grasp the situation presented. We may compare the speed of the reader of a short story to the speed of a car down a hill or on level road. The speed of a reader of a problem in mathematics may be compared to the speed of a truck in low gear on a hill. More important than the speed of reading is the element of comprehension and understanding of concepts. In reading a verbal problem, the student must grasp the situation presented, be fully aware of the meaning of technical terms and quantities, and be able to discover relationships among numbers. This special form of reading is limited almost exclusively to the study and interpretation of numerical and quantitative data which are expressed in either tabular or verbal form.

Teaching Reading Skills Needed in Arithmetic

Success in arithmetic and other branches of mathematics is closely associated with proficiency in the special types of reading skills necessary in this field. This is particularly true in the reading of explanations of procedures given in textbooks, in problem solving, and in the gathering of information of materials from printed sources for any of a number of different purposes. The analysis of uses of reading in mathematics that is given below should be read carefully by the teacher to get an overview of this important phase of instruction.

READING SKILLS REQUIRED IN MATHEMATICS COURSES

1. Reading of numbers and comprehension of their meaning.
   a. The ten numerals, 1, 2, 3, 4, 5, 6, 7, 8, 9, 0.
   b. Place value in Arabic numbers, such as 327.
   c. Meaning of 0 in such numbers as 20, 0.6, and 301.
   d. Meaning of common and decimal fractions.
   e. Numbers on mechanical computation devices.
   f. General and signed numbers, as -4, a^2, $\sqrt{6b}$.

2. Reading related to number operations.
   a. Symbols of operations, such as +, −, ×, etc.
   b. Meaning of numbers as they are manipulated in each step of an operation, for example, in adding 463 and 137.
   c. Reading of explanations of process procedure in a textbook.
   d. Reading of directions about assignments included in textbook.
   e. Reading equations.
3. Vocabulary of mathematics.
   a. Meaning of technical mathematical terms, such as digit, add, hundred, area, formula.
   b. Meaning of units of measure (also abbreviations and symbols).
   c. Meaning of the quantitative vocabulary related to social applications of arithmetic, such as stamp, amount of taxation.
   d. Use of dictionary, glossary, etc., to get definitions.
4. Basic skills involved in reading and solving verbal problems.
   a. Comprehension of the meaning of items and statements contained in the problem.
   b. Reading necessary to apply the steps usually taken in problem solving which are:
      (1) What are you asked to find?
      (2) What facts are given? Is other information needed?
      (3) What steps must be taken to solve the problem? (It is necessary to see the relations between the parts of a problem to determine steps.)
      (4) What would be a reasonable answer?
   c. Locating information not stated in the problem but necessary for its solution.
      (1) In accompanying tables, graphs, charts, pictures, etc.
      (2) In preceding problems and discussions.
      (3) In reference books, catalogs, and other printed sources.
      (4) In appendix.
      (5) In schedules, forms, plans, maps, etc.
   d. Reading formulas, equations, rules.
5. Reading and using quantitative aids and measuring devices.
   a. All numbers needed are given, as in the calendar.
   b. Interpolation required in reading a scale, as in thermometer.
   c. Fractional interpolation may be involved, as in reading the ruler.
   d. Enrichment of meanings because of awareness of the history and social significance of devices used in measurement, such as money, clocks, protractors, or thermometers.
6. The reading and interpretation of statistical tables of a wide variety of kinds.
   a. Reading standard tables, such as interest tables and trigonometric ratios.

b. Reading tables presenting factual information of a social nature.
   (1) Identifying the contents of the table.
   (2) Understanding the structural elements and arrangement of the table.
   (3) Selecting detailed information included in the table.
   (4) Interpretation of information included
   (5) Summarizing information secured.
   (6) Evaluating the information included.
   (7) Making generalizations and comparisons based on the data.
   (8) Predicting future happenings on the basis of the information given in the table.
   (9) Remembering information.

7. The reading and interpretation of graphs.
   a. The comprehension of graphs of various kinds, such as bar graphs, circle graphs, histograms, etc.
   b. Reading skills similar to those listed under 6-b above are involved in the reading of graphs.
8. The interpretation of quantitative elements included in diagrams, charts, maps, plans, and pictures.
   a. Interpretation of quantitative symbols on charts, maps, etc.
   b. Reading of scales, and scale drawings.
   c. Longitude and latitude readings on maps and globes.
   d. Pictograms.
   e. Use of guide lines to locate information, places, etc
   f. Comprehending quantitative concepts included in advertisements, business forms, pictures, etc.
9. Reading skills involved in securing information about assignments in the study of social applications of mathematics such as:
   a. Locating information in various printed sources.
   b. Comprehending what is read.
   c. Organizing what is read, as making an outline of contents.
   d. Evaluating what is read.
   e. Summarizing what is read.
   f. Making generalizations.
   g. Remembering what is read.

The list shows that a student needs a wide range of reading skills to succeed in mathematics. To read a verbal problem intelligently, the student should be able to apply the steps usually taken in problem solving which are listed under 4-b in the above outline of reading skills. First, he should clearly identify the key question for which the answer is to be found. Second, he should

identify the facts that are given. And third, he should discover dependencies or relationships between what is given and what is to be found. A verbal problem always involves finding some missing quantity or using a concept or quantity in a given situation. The third factor pertains to the relationships expressed or implied in the problem. The situations given must support certain dependencies among the facts given or the student cannot solve the problem. We may illustrate these three elements of a verbal problem by considering the following problem:

> An airplane travels 200 miles in 40 minutes. At that rate, how far would the airplane travel in one hour?

The student should identify the question for which the answer is to be found. In this case it is the distance the airplane can travel in one hour. Next, he should identify the data that are given, namely, the distance the airplane can travel in a fixed interval of time as stated. Finally, he should sense that the distance an airplane will travel in an hour depends upon the following chain of events:

1. The distance an airplane can fly in an hour depends upon how far it can fly in a fractional part of an hour or a given number of minutes.

2. The distance an airplane can fly in one minute depends upon how far it can fly in the given time.

3. The distance an airplane would fly in an hour depends upon how far it travels in one minute and the number of minutes in one hour.

From this analysis it is seen that a student must have a background which would enable him to discover the relationship between speed per hour and the distance traveled in a given interval of time. The plan outlined above may be designated the method of unitary analysis.

If the student is able to discover that the problem calls for finding a number when a fractional part of it is given, he would be able to give a quicker solution than the solution by analysis. If the airplane travels 200 miles in $\frac{2}{3}$ hour, the problem may be expressed as

$$\frac{2}{3} \times ? = 200$$

The missing number is found by dividing 200 by $\frac{2}{3}$ which is the same as multiplying 200 by $\frac{3}{2}$. In this solution the student identifies the problem as one which fits the pattern of finding a number when a fractional part of the number is given. He has learned the basic principle that when the product of two numbers and one of the numbers are known, the missing number is equal to the quotient of the product divided by the known number. Regardless of the method used, the student must have a background which enables him to discover relationships among quantities and to know how one item depends upon another essential item or element in the problem situation.

Logical Pattern of Problem Solving

The logical approach to teaching problem solving is familiar to most teachers of mathematics. According to this plan, the teacher should instruct the student to follow the plan outlined below:

1. Find what the problem question is.
2. Then find what facts the problem gives.
3. Try to think of ways to find the answer to the question asked, or search for a familiar pattern or model in the problem.
4. Do the necessary computation.
5. Check the answer to see if it is sensible.

This logical pattern has merit for a disciplined mind, but immature students do not follow such a logical pattern in their thinking. Several investigations[5] show that a logical structural pattern is not an effective means of teaching problem solving. Fehr expressed the conditions under which problem solving takes place as follows: "Real problem solving involves seeing the problem as a whole, familiarity with all of the elements of the problem situation, making analyses, seeing relations, getting a pattern of relationships, estimating and checking, and reorganization. We seldom solve real problems in organized deductive steps. This

[5] See the following study as representative of this group: Hanna, Paul R. *Arithmetic Problem Solving: A Study of the Relative Effectiveness of Three Methods of Problem Solving.* New York: Bureau of Publications, Teachers College, Columbia University, 1929.

deductive structure is usually made after insight to the solution has been made."[6] Fehr is not discussing problem solving as represented at levels I and II. When problem solving is taught at these lower levels, the role of insight usually plays a less vital part than it plays in dealing with a real quantitative situation.

The student who has a sufficient background to be able to discover relationships among quantities usually devises an individual method of solving problems. Before he is able to develop an effective individual method, he must discover a general approach to use in attacking a problem. It is for this reason that the slow learner can profit by observing the following three steps in reading a problem:

1. Find what the problem question is.
2. Identify the information and numbers given in the situation the problem presents.
3. Show how the answer to the question asked depends upon what is given.

The student should be taught to search in the problem for a familiar pattern which he can use to arrive at the solution. The fact that a slow learner is able to read a problem and identify what question is involved in the problem helps him to feel secure. Many slow learners read words which are devoid of meaning to them. Problems should be expressed in the simplest language so that the less competent student is able to read them. Instead of trying to solve problems typical of those given in the seventh grade, slow learners should read problems suitable for students who are one or more grades below this grade level. Often these students need special help in reading problems of the type that can be found in textbooks and workbooks for lower grades or in other supplementary materials. The student may not be able to cope with the problems in his textbook at his grade level. He then should be given similar materials of a lower level of difficulty than the regular textbook. After he experiences success with the easier problems, he should attempt the types found in his own textbook.

[6] Fehr, Howard F. "The Role of Insight in the Learning of Mathematics," *The Mathematics Teacher*. 47:392.

Easily-solved Problems

Students at the junior high school level should solve many verbal problems which are predominantly one-step problems so that the dependency between the question asked and the facts given is easily determined. These problems also should contain small numbers so that the computation involved will be easy. Samples of this type of problem suitable for Grades 7 and 8 are the following:

1. What is the cost of 1000 3-cent stamps?
2. How many pints are there in 4 gallons?
3. At an average speed of 50 m.p.h., how many minutes does it take to travel 25 miles?
4. A roll of dimes has a value of ten dollars. How many dimes are in the roll?
5. Grapefruit sell 3 for 25¢. At that rate, what is the cost of a dozen grapefruit?
6. What is the smallest three-place number which is divisible by 3?

Many of the superior students at the junior high school level are able to solve problems of the types given without the use of paper and pencil. These students should be urged to solve such problems "in their heads." This form of arithmetic is sometimes known as "mental arithmetic." Other students often must use paper and pencil. The slow learners may in addition have to make visual representations or use objective materials to solve the problems. Since there are no complex relationships among the quantities in these problems and the computations are easy, these problems offer excellent practice in reading and solving verbal problems of a wide variety of patterns.

Mental Arithmetic

Hall defined "mental arithmetic" as solving problems that arise in an oral manner or "in the head" and the solution is given without the use of paper and pencil. From a limited survey of business usage, he found that the majority of arithmetic problems

arising in life are solved mentally. Hall suggested that the following values may be obtained from mental arithmetic:

1. Mental arithmetic emphasizes the importance of place value and the "ten-ness" of our number system
2. It brings number relationships into focus; this insures understanding in addition to saving time
3. It serves to find approximate answers to arithmetic problems since, in life, arithmetic is not concerned exclusively with computation
4. It helps children to direct their attention immediately to the conditions of the problems, since there is little or no writing necessary

. . . .

5. It enables the teacher to gain an understanding of the child's quantitative thinking, if he is encouraged to explain his solution of a problem orally to the teacher
6. It is valuable in introducing new arithmetic processes at all grade levels because the child's attention can be focused on one step at a time
7. It serves to enrich the regular program of numbers when used as the basis for number quiz games.[7]

From the result of an experimental study dealing with solving problems without use of pencil and paper, Petty[8] suggested that teachers should set aside certain pages of verbal problems in the textbook which would be solved "mentally."

Variation in Ability in Problem Solving

The range among students in ability to solve problems listed under levels I and II is very large. There are two ways in which all teachers can adjust instruction to these variations. First, superior students should be expected to apply procedures that are superior both qualitatively and quantitatively to those used by the group below average in problem solving. Second, the superior students often are able to assist some of the slow learners in learning to solve problems.

[7] Hall, Jack V. *Solving Arithmetic Problems Mentally*, pp. 9–10 Educational Service Publications, No. 20. Cedar Falls: Bureau of Exterior Service, Iowa State Teachers College, 1954.

[8] Petty, Olan "Non-Pencil and Paper Solution of Problems," *The Arithmetic Teacher*. 3:235.


Santa Monica (California) Unified School District   Photo by Earl Dible

When fast learners assist others, often all benefit from the experience.

The quantitative difference in abilities of students refers to the number of problems to be solved by the superior group. This is a minor factor. Frequently teachers place undue emphasis on the number of problems or examples a student should solve to provide for individual differences. Superior students are thus assigned extra practice that they do not need. This use of their time is of little value.

The qualitative difference refers to the type of problem and the kind of solution given. The superior student should solve more difficult problems than the students less gifted in dealing with quantities and operations. Many textbooks contain starred problems and pages of starred problems which are more difficult to solve than those found in the regular body of the text. The superior student should be able to solve problems in this category.

The qualitative differences between groups can further be demonstrated by the kinds of solutions given to a problem. The teacher should give students the opportunity to suggest a variety of solutions to problems. The teacher who has a fetish for regimentation demands that all solutions to a problem must be the same. Students should be stimulated to give a variety of different solutions to a problem. Fehr stated that ". . it is better to examine and compare several different solutions to the same problem than to solve three similar problems by the same method."[9] The solutions given below to the following problem illustrate different ways students may find the answer:

A car travels 5 miles in 6 minutes. At that rate, what is its speed per hour?

a. The car travels $\frac{5}{6}$ mile in a minute. In 60 minutes the car would travel $\frac{5}{6}$ of 60 miles, or 50 miles.

b. There are 10 6-minute periods in an hour. Then, $10 \times 5 =$ 50, number of miles in an hour.

c.

5	miles in	6	minutes	
10	" "	12	"	✓
20	" "	24	"	
40	" "	48	"	✓

The sum of the numbers checked will be the distance traveled in 1 hour. 10 mi. + 40 mi. = 50 mi.

The three methods illustrate different solutions to the problem. Others can be devised. The teacher will be amazed at the wide variety of solutions which students give when they are encouraged to use exploratory methods. The teacher must be certain that all of the solutions accepted are valid.

d. Teaching Problem Solving at Levels I and II

Basic Instructional Procedures to Use

The methods of procedure for teaching problems listed under levels I and II are similar. In each case a problem is defined as a verbal problem. At level I, the problems are miscellaneous in nature and deal primarily with unrelated topics and processes,

[9] Fehr, Howard F. *op. cit.*, p. 391.

but at level II, the problems deal with some particular topic or phase of mathematics, such as per cent, area, or volume. It should be understood that problems at these two levels do not afford opportunities for quantitative thinking comparable to the problems which characterize those classified under the other two levels.

Kinds of Helps Needed at These Levels

Students who read problems well and have good backgrounds for interpreting quantitative statements should encounter a minimum of difficulty in solving verbal problems found in representative textbooks in mathematics in the junior high school. To help students to solve such problems, the teacher should ask questions which will enable those who have difficulty in solving the problems to discover a relationship between the facts given and the missing quantity to be found. We may illustrate the questioning technique to use in helping students to solve the following problems:

1. A boy buys apples at the rate of 3 for 5¢ and sells them at the rate of 2 for 5¢. How many apples must he sell to make a dollar?

To help the student understand this problem, the teacher should ask questions such as the following:

"What does the problem ask you to find?"

"Does 'make a dollar' mean that the apples are to be sold for a dollar? Are they to be bought for a dollar? What does it mean?"

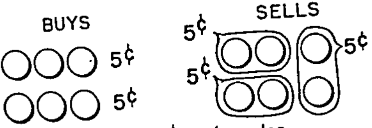
"Do we know how many apples he should buy or sell?"

"Do we know the cost of one apple? Can we find the cost of one apple? the selling price of one apple?"

"If we know the cost of one apple and the selling price of one apple, how can we find the gain made on one apple?"

"If you know the gain made on one apple, how do you find the number of apples needed to gain a dollar?"

"If you know the cost of an article and its selling price, how do you find the number of those articles to be sold to gain a certain amount?"



"We had to use fractions to find the answer to the problem. Let us represent the cost of 3 apples as shown." Now the teacher has the student use markers or some other objective material to solve the problem. He continues to use these objective materials until he discovers a relationship between the gain or profit made on 6 apples, or a multiple of 6, which will enable him to find the answer to the problem.

2. The perimeter of a square is 80 feet. What is the area of the square?

"What is the problem question?"

"What must be known to find the area of a square? Is this fact given? What fact is given?"

"What does the word perimeter mean? Make a drawing of the square to show the meaning of the term perimeter."

"Use this carpenter's rule to represent the perimeter of a square or a rectangle."

"The perimeter of a square is how many times as long as a side? If we know the perimeter of a square, how do we find a side?"

"What does the term area mean?"

"On the square you draw, show its perimeter and its area."

"If you know the side of a square, how do you find its area?"

"Why should the area be expressed in square feet when the perimeter is expressed in feet?"

The questions given above for the two problems are representative of the kinds of questions the teacher should ask to help the student to discover a solution to a verbal problem which he cannot solve. Special attention must be given to see that a student understands the meaning of the terms used in the problem and that he can determine what the problem asks. Then he must be able to see how the question asked depends upon the facts given.

The teacher must give help and guidance to the student to enable him to discover relationships among quantities. Craig has summarized this point well in the following statement: "The more guidance a learner receives, the more efficient his discovery will be; the more efficient discovery is, the more learning and transfer will occur."[10]

To repeat, problem-solving ability cannot be developed to a given level as a skill, such as attaining a certain proficiency in computation. The student's ability to solve problems depends upon his experience and his background. The special helps pertaining to vocabulary, reading, and procedures for dealing with verbal problems are given in this text in order to help teachers do as good a job as possible with this low level of problem solving.

Labeling the Answer

The problem of how to label answers is one which all teachers of mathematics encounter. Should a student have his solution marked wrong if he gives 400 instead of 400 square feet as the answer to problem 2 on page 326? Clearly an answer of 400 is not complete. Soluion A on the right shows the form to use. Solution B is not acceptable. On the whole the student should work with unlabeled numbers, except with money. Then he should copy the answer with its label as shown in A. Ullrich[11] reported the results of a questionnaire sent to 275 teachers, school administrators, and authors of textbooks in arithmetic. The study showed that there is great variation in practice and opinion about the correct labeling of answers in arithmetic.

A	B
$\begin{array}{r} 20 \\ \times 20 \\ \hline 400 \end{array}$	$\begin{array}{r} 20 \text{ ft.} \\ \times 20 \text{ ft.} \\ \hline 400 \text{ sq. ft.} \end{array}$
A = 400 sq. ft.	

[10] Craig, Robert C. *The Transfer Value of Guided Learning*, p. 72. New York: Bureau of Publications, Teachers College, Columbia University, 1953.

[11] Ullrich, Anna M. "Report on the Investigation Concerning the Marking of Answers to Problems in Elementary School Arithmetic," *The Mathematics Teacher*, 46:292–293.

e. Solving Problems Classified as Level III

Types of Problems at Level III

Problems classified as level III are related to and deal with a central theme. These problems may be based on a described situation, a table, a graph, or information pertaining to a particular topic.

Page 329 shows a page from a textbook. The problems on that page are based on the data in the table. The teacher should ask the class pertinent questions about the table before assigning the problems for solution. If the students are able to answer the questions, the teacher can be reasonably certain that the students understand the table. The teacher should ask questions of the type which follow:

1. What is the title of the topic we are to consider?

2. The picture shows a corn-cutter and husker. How many of you know what a corn-cutter and husker is and where it is used?

(Students who live in a corn belt area would know about a corn-cutter and husker, but students in an urban area probably would not know. If the class is not familiar with the term, the teacher should have a student find its meaning in a dictionary or an encyclopedia. Then the class should discuss some of the other means of harvesting grain. A committee could be appointed to report at the next class period on different ways used in this country to harvest grain. The committee should report on such terms as sickle, cradle, reaper, binder, and combine.)

3. What are some of the reasons given for increase in production of crops by farmers? Where can you find the answer to this question? (The teacher should be sure that the student understands the first sentence on the page.)

4. Several weeks ago we found on page 113 how the use of fertilizer increased the yield of corn per acre. (The class should turn to this page and review how the use of fertilizer will affect the yield per acre. The students should now understand how crop production depends upon the use of fertilizers.)

5. The table shows the average number of persons that can be fed from the production of an average farmer. The table

Eleanor Rast from Frederic Lewis

Farmers Grow Bigger and Better Crops

The wider use of farm machinery, new fertilizers, and selected seeds has enabled farmers to increase the yield of crops. The table gives the number of people that could be fed with the crops produced by the average farmer.

Year	Number of People	Year	Number of People	Year	Number of People
1850	4.8	1890	7.2	1930	11.0
1860	5.1	1900	8.1	1940	11.3
1870	5.8	1910	8.0	1950	15.0
1880	6.4	1920	9.9		

1. The average farmer in 1950 produced enough food to feed approximately how many times as many people as the average farmer in 1850?

2. Between which two 10-year entries in the order given was there the greatest increase in the production of food?

3. Between which two 10-year entries in the order given was there a decline in the production of food?

4. A million farmers were able to produce enough food to feed how many people in 1850? in 1900? in 1940? in 1950?

5. Our population in 1950 was 150 million. It took at least how many farmers to produce enough food to feed our people?

6. It is estimated that 18% of our population are farmers. If our population is 160 million, how many farmers are there?

★7. Make a line graph or a bar graph to show the data in the table.

covers a period of how many years? What is the time interval from one entry to the next entry?

6. In 1850 the average farmer could produce food to feed an average of 4.8 persons. How is it possible to have 4.8 persons? We can approximate 4.8 as what number?

7. If we round off the number in the table for 1850 to be 5, in which year did the average farmer produce enough food to feed about twice as many persons as in 1850? three times as many persons as in 1850?

8. The average farmer in 1950 produced enough food to feed an average of 15 persons. Does this mean that each farmer produced enough food to feed 15 persons? Give several reasons why some farmers would produce more food than other farmers.

9. Using the same time interval given in the table, what year would be the next entry? Would you predict that the average number of persons fed per farmer for that year would be more or less than 15? Why?

After the class has discussed questions of the type given, the teacher should assign the problems on the page for solution. All students should be able to solve problems which are based directly on the reading of the quantities in the table. Every student should be able to find the number of people who could be fed by the crops produced by a certain number of farmers, such as a million, in one of the given years. The discussion enables the student to acquire a background for intelligent reading of the problems.

Problem 6 states that farmers comprise 18 per cent of our population. This fact with certain data in the table can be used to show that farmers can produce more food than can be consumed by our population. The teacher should assign to the superior students the topic of crop control and have them find out what crop control means and what arguments can be given for and against it. From the standpoint of the federal government, the problem of the disposition of surplus crops constitutes a basic problem of agriculture. The superior students should investigate the topic of crop surplus and the need for crop control as shown by the table. A report of this kind should be of interest to the whole class and should help the students to understand

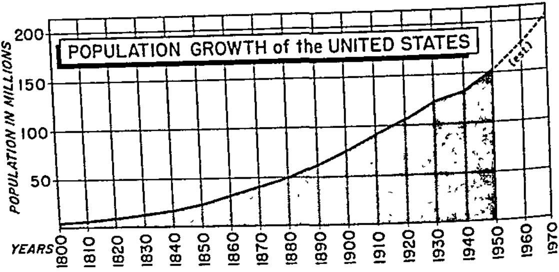
something about one of the major internal problems with which our government must deal.

Teaching Students to Read Graphs

A graph is a visual representation of tabular data. A graph should enable the reader to see number relationships among quantities much more easily than when the data are given in a table. Since problem solving involves dealing with relationships among quantities, graphic visual representations, which express relationship among quantities, are elements of a program for teaching problem solving.

There are two phases of reading a graph. The one phase consists in understanding the mechanical features of a graph, while the other phase consists in interpreting the data. The student should know how to read a graph and be able to determine whether or not it is drawn properly so as to portray the data correctly. The ability to read a graph and to criticize it from the standpoint of physical features constitutes a skill. The interpretation of the data in a graph depends upon the background of the individual. He may be able to read a graph correctly, but he may not be able to interpret the meaning of the data because he is not familiar with the topic represented.

The function of a graph is to present data in a more easily interpreted form than tabular form. At the junior high school level, a mathematics textbook should provide instruction in reading and drawing graphs just as it provides instruction in a basic number process. The student should refer to the graphs provided in his textbooks to become more familiar with methods of discovering relationships among quantities. In that sense a graph is an aid in problem solving. Unfortunately, many mathematics textbooks contain units which show the student how to read and draw graphs and then make no further reference to graphs. The chief functional value of a graph is lost in a presentation of this kind. A graph is a visual aid to help the student grow in ability to deal with relationships among quantities. This statement is true not only in mathematics but also in other areas of the curriculum. The student should learn to



read graphs in order to interpret the data given in the social sciences or in other subjects he may be studying.

The above graph shows the growth of our population from 1800 until the present time and the predicted population in 1970. The student should first note that time is represented on the horizontal scale. The time interval is uniform, but the increase in population is not. A line graph has two variables which in this case are time and population. The horizontal scale is used to represent the variable which changes at a uniform rate.

The student must understand that numbers read from a graph are predominantly round numbers. In the graph above, the population in 1880 was approximately 50 million and in 1900 it was approximately 75 million. From the graph it is not possible to state exact values for a given year, but the values read are close approximations of the true value. The use of round numbers simplifies the work of dealing intelligently with large quantities.

A line graph is difficult to read. Slow learners should not be expected to develop skill in reading line graphs. Most certainly the reading of bar graphs and pictograms should be part of the curriculum for all students at the junior high school level, but the reading of line graphs should not be part of the basic work for all of them.

gives the value of a point on a graph between two known points. He extrapolates when he projects the graph to some point beyond the last given value. The reader may be asked to give the reason for a marked variation or deviation in the trend of the graph, as the change in the curve of our population during the decade from 1930–39 and the big increase in the trend after the close of World War II. Thus, it is seen that the kind of information derived from a graph may vary from a factual to a highly interpretative type. It should be evident that only the very superior student will be able to answer interpretative questions based on graphs.

Appraising Graphs

A student should be as sensitive to an error in a graph drawn incorrectly as he should be to a grammatical error in English. In some graphs there are certain types of glaring errors which the student should be able to identify. Some of the principles of graph construction which frequently are violated are described below.

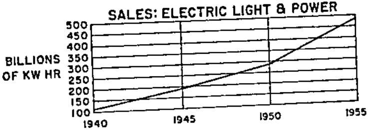
1. A picture graph should not be interpreted in two dimensions. If a manikin represents quantity A and quantity B is twice as large as A, two manikins of the same size as the manikin in A should represent B. If the manikin for B is twice as high and twice as wide as for A, the area represented by manikin B would be four times the area for manikin A.

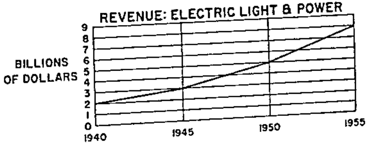
2. Graphs usually are more accurate when the vertical scale begins at zero. If this scale does not begin at zero, the graph should show that part of the scale was cut off. The usual way to denote this fact is to use two wavy lines.

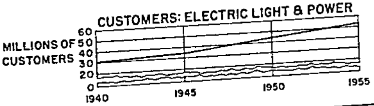
3. The scale should be uniform. If time is represented, there should be a regular interval between consecutive points on the scale.

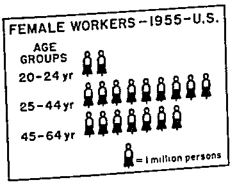
4. The graphs should be well drawn, easy to read, and properly labeled.

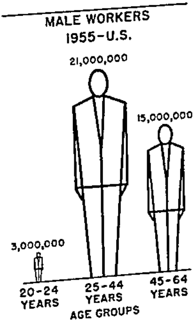
Some graphs on page 335 represent well drawn picture graphs, bar graphs, and line graphs and there is at least one sample of each type of violation mentioned above. The reader should be

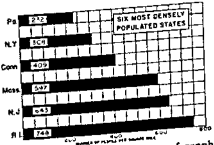












Correct and incorrect techniques of graph making. (See page 334.)

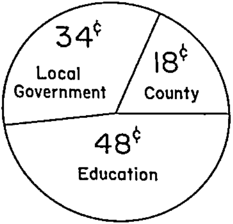
able to identify each type of violation of graphic technique. If a student can detect an error in the drawing of a graph so that the visual representation does not present an accurate picture of the data, he has learned to do critical thinking in dealing with quantities. Development of that ability is one of the functions of problem solving.

f. Solving Problems Classified as Level IV

Sources of Real Problems

Problems classified as level IV grow out of the activities of the students or of the class. These problems emerge in the experience the student may have in some social situation in the community or the classroom. A few illustrations of the kind of activity involved will help clarify this type of problem.

A class had studied about property taxation as discussed in the textbook. Under the guidance of the teacher, the group decided that it would be profitable to find out about the local tax rate. The students brought to class samples of tax bills for the past few years. Having learned that the town or city was the local unit of government in the state, the students secured the tax rates existing in their own community and those in a few neighboring communities. Next, the class invited a man from the tax bureau to speak to them. He gave them information about how property in the town is assessed, how the tax funds are allocated to local and county governments. The students were told how the funds for local government are divided between education and other local governmental functions. Using these data, the students made circle graphs as shown indicating how the tax dollar is spent for local purposes.



The LOCAL TAX DOLLAR

A few days later a member of a commercial bank in the town spoke to the class about his bank. He showed samples of the most familiar banking forms used for making deposits, various kinds of checks, and forms for making a loan. He explained the difference between personal and secured loans and how each is made. Finally, he arranged a conducted tour of his bank for the students to whom he had spoken.

As often happens in a study of topics having great social significance, most of the mathematics involved is informational. There is a limited amount of problem solving in the narrow sense of the term, in dealing with topics that are predominantly informational. On the other hand, the student learns the meaning of certain concepts from experiencing them in functional situations. Then, too, he learns the functions of the basic processes. If a test reveals deficiencies in skills, the student's discovery of the use of number in real life problems should motivate him to develop these skills.

Clifford reported the result of a survey made by junior high school students in a local community about the different procedures in computing interest on a savings account which had different balances during an interest period. The students found that four different procedures were used in determining the balance on which interest would be paid after deposits or withdrawals were made during each interest period of six months. A summary of the findings was as follows:

Bank A credited interest on the minimum balance during the interest period.

Bank B divided the half year into two quarters and used the minimum in each quarter for computing interest.

Bank C gave interest on the minimum balance plus interest on deposits for each full month that the deposits were credited during the interest period.

Bank D computed interest from the day of deposit.

Clifford concluded as follows: "The actual procedures varied even more than this summary indicates. Interest rates varied from 1% to $2\frac{1}{2}$%. Some banks counted actual time, others used approximate time. In several banks the only person who would take the responsibility of explaining their system was a vice-

president, which would indicate that explanations are requested very seldom."[12]

An investigation of this kind gives the student first-hand experience with the operation of one of the most familiar institutions in a community. The results of a study similarly conducted frequently will show the investigator that banks are not uniform in methods of paying interest on savings accounts. In case of commercial banks, the cost of maintaining a personal checking account is not uniform among these banks. The student who has knowledge of many of these variations among banks has received very useful consumer education.

Solving Problems that Cannot Be Answered with Finality

A student should have some experience with real problems which cannot be answered with finality. The following problem is in this category: Is it cheaper to rent or to own a home? Students can collect data which bear on this problem, but probably it would be difficult in light of the facts gathered to determine which plan is the cheaper.

A college student who was recently married and needed housing service brought to class the following problem:

A house is offered at rental of $100 a month for a 3-year lease, or $90 a month if paid in one sum in advance. Which is better?

The class discussed various factors which affected the problem. One of the students thought it would be easier to break the lease if the money were not paid in advance. This element was finally ruled out as the contract would be equally effective under each plan. The students admitted the problem was real and one which was vital to a family in need of housing. They computed the rate of interest charged on a loan made under the stated conditions. The problem was an excellent illustration of an instalment loan in which a month's rental corresponds to the down payment on an article purchased on the instalment plan. By using the conventional formula, $r = \frac{24\,c}{p\,(n+1)}$, for computing the rate of an

[12] Clifford, Paul C. "A Simple Matter of Interest," *The Mathematics Teacher*, 47.28.

Public Schools, Montclair, New Jersey

The school bank often provides real life problems involving arithmetic.

instalment loan, the students found the rate to be approximately 7.4 per cent. They recognized that this was a high rate of interest for a loan, but other factors had to be considered besides the rate. One student's remarks were significant. "It would be cheaper to pay the amount in advance, but where are you going to get $3240 for one payment? You must pay more to have the privilege of monthly payments." The students used both informational and computational arithmetic to arrive at an answer to the problem. They dealt with quantities and experiences and exercised a high level of critical thinking about these quantities.

Questions, Problems, and Topics for Discussion

1. How do you define a problem? Give an illustration of a real life problem with which you were faced and state the ways in which mathematics helped you to solve it.

2. Make a list of urges or motives which might stimulate a student to solve a problem. Evaluate the different kinds of motives which impelled you to solve real life problems.

3. When you meet an unfamiliar problem in arithmetic or some other branch of mathematics, how do you go about finding a solution to it? Illustrate.

4. A noted psychologist related the story of an elementary school pupil who used the following procedure for selecting the process to use in solving verbal problems: "If the problem has three numbers, I add; if it has two numbers about the same size, I subtract; if one number is large and one small, I divide

to see if they come out even; if they don't come out even, I multiply." From the standpoint of discovery, appraise this plan which the pupil devised. Evaluate the program of teaching problem solving in the school in which this pupil was enrolled.

5. Two trains traveling in opposite directions from a given point have speeds of 45 m.p.h. and 60 m.p.h., respectively Make a drawing to show how far apart these trains would be at the end of 3 hours.

6. Make an analysis of the problems in an arithmetic textbook for the seventh or eighth grade and find approximately what per cent of the pages of problems may be classified at each of the three levels discussed in this chapter.

7. Make a list of topics which are adaptable for investigations involving problems classified in this chapter at level IV

8. "The best way to teach problem solving is to give the student plenty of problems to solve," said a teacher. Evaluate this statement. Does that teacher hold to a quantitative or qualitative concept of the worth of problems?

9. Show how you would differentiate the curriculum in problem solving to provide for individual differences of students.

10. Write approximately a dozen quick-answer problems which are suitable for students at the junior high school level

11. Evaluate the following statement. "It is not possible to teach problem solving *per se*."

12. What are some of the concepts in the following problem which might make the solution difficult for a student in the eighth grade? Show how you would proceed to teach a student to solve this problem.

Find the premium on a life insurance policy of $7500 at a rate of $34 50 per $1000.

13. A car travels 75 miles in 2 hours. At that rate, how long will it take the car to travel 300 miles? Give at least three different solutions to this problem.

14. A student wrote the answers given to the following problems.

a. How many inches are there in 3 feet? 36

b. Find the area of a rectangle 4′ × 6′ 24

How would you mark his answers? Under what conditions do you want the answer labeled correctly?

15. If a student uses the correct method in solving a problem but makes a computational error in finding the answer, do you mark the problem right, wrong, or do you give partial credit for the correct method? What do standard tests do?

16 Do you permit a student to use a textbook when he takes a test in problem solving? Should you?

17. Suppose you have a student who confused the concepts of area and perimeter. How would you help him to understand the difference between these two terms?

18. Bring illustrations of graphs to class and evaluate them from the standpoint of accuracy of drawing and ease of reading

19. Make a list of at least ten mathematical statements to show how one thing depends upon another thing. For example, the cost of a given number of articles depends upon the price of one article and the number of articles.

20. An airplane has an average speed on an outgoing flight of 100 m.p.h., but on the incoming flight the average speed is 150 m.p.h. What is the average speed on the total trip?

If you solved the problem as many students do, you found the average of 100 and 150 which is 125. Select some distance for the flight, as 600 miles, and prove that 125 m.p.h. is not the average speed of the airplane. Now analyze your thought pattern. What was the cue which caused you to give the incorrect solution? From your experience, write a generalization which pertains to the averaging of speeds. (The average speed of the airplane is 120 m.p.h.)

Suggested Readings

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Chapter 10

Ratio, Proportion, and Square Root

This chapter deals with the following topics:

a. The use of ratio in solving problems
b. Teaching how to solve a proportion
c. The meaning and use of similar figures
d. Methods of applying indirect measurement
e. Finding distances by using the tangent ratio
f. Square root and its applications.

a. The Use of Ratio in Solving Problems

How to Find the Ratio of Two Numbers

Two numbers can be compared either by subtraction or by division. If John has \$4 and Dick has \$5, we may say that John has one dollar less than Dick, or that John has $\frac{4}{5}$ as much money as Dick. The fraction, $\frac{4}{5}$, formed by dividing \$4 by \$5, expresses the *ratio* of the two amounts. The fraction $\frac{4}{5}$ may be expressed as .8, or 80 per cent. Similarly, the ratio of Dick's amount of money to John's amount may be expressed as $\frac{5}{4}$. If Tom has \$8 and Bill has \$4, then the ratio of the two amounts is $\frac{8}{4}$, or 2. We see that the ratio of two amounts may be expressed as a proper fraction, an improper fraction, an integer, a decimal, or as a per cent.

There are three ways of expressing the ratio of two numbers:

$$4 \div 5 \qquad \frac{4}{5} \qquad 4:5$$

Since a fraction represents an indicated division, it is easy to see that the two forms $4 \div 5$ and $\frac{4}{5}$ *are the same*. The expression 4:5 is read, "Four is to five" and it means the ratio of 4 to 5. This form is seldom used. The fractional notation is the most usable way of expressing a ratio.

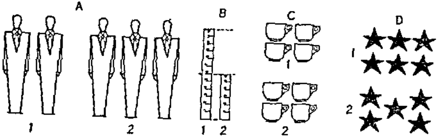
Steps in Teaching Meaning of Ratio

The following steps are recommended for developing a student's understanding of ratio so that he will be able to deal intelligently with this very important concept:

1. The teacher should provide each student with disks or markers to enable him to demonstrate the ratio of two numbers. Then the teacher should introduce the work with an easy problem, such as comparing the number of school days in a week with the number of calendar days in a week. Each student should put 5 disks in one stack and 7 disks in another stack to represent these two numbers. Then he should express the ratio of the height of the smaller stack to the height of the larger stack as $\frac{5}{7}$. Similarly, he should express the ratio in the reverse order as $\frac{7}{5}$. He should use disks to show the relationship between other pairs of unequal groups until he is able to discover that the fractions expressing the ratios of two quantities are *reciprocals* of each other. Thus, the fraction, $\frac{5}{7}$, is the reciprocal of the fraction, $\frac{7}{5}$.

If the term reciprocal is not used, the term inverted fraction may be substituted for reciprocal. Of course the same principle applies when the ratio of the quantities is a whole number. If the number of disks or chips in one stack is three times the number in another stack, the ratio of the two quantities is 3 to 1, or 3. The reciprocal relationship of the heights of the two stacks is $\frac{1}{3}$.

The comparison of stacks of chips can also be used to objectify the meaning of the ratio of two equal quantities. In this case the ratio of the two compared quantities is 1. The teacher should have the student interpret such a familiar expression as "fifty-fifty" to mean that the groups are equal and that the ratio of the amounts involved is 1.

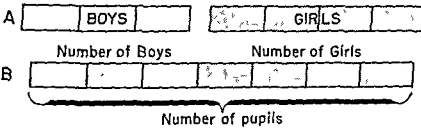


2. The student should express the ratio of two quantities which are represented visually. The diagram shows the type of representation to supply at this stage in the development of the topic. In each case the student should express the reciprocal relationship between the two quantities represented in pictorial form.

3. The student should make drawings to express the ratio of two numbers. Thus, to show the ratio of 3 boys to 4 girls, he would make a drawing of the type shown in A below.

In B, the student should discover that the number of boys is $\frac{3}{7}$ of the total number of pupils, and that the number of girls is $\frac{4}{7}$ of the total number of pupils.

4. The student should use disks, drawings, or both to represent two quantities when the ratio of one quantity to the other quantity is given. Thus, if the ratio of the number of boys to the number of girls in a class is $\frac{3}{4}$, each student should show this ratio with a diagram of type A given in step 3. Then the class *should discuss the meaning of each part of the diagram used to* represent the two groups. The diagram should show that for every three boys there would be four girls. Each rectangular part of the diagram would represent one student, or any number of students. If each part represents two students, then for each 6



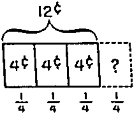
boys there would be 8 girls. This fact can be expressed as $\frac{3}{4} = \frac{6}{8}$. When the student knows how to represent graphically a ratio of two quantities, he has acquired a skill of great usefulness in solving problems in which it is necessary to find a number when a fractional part of it is given.

b. Teaching How to Solve a Proportion

The last step in the development of the ratio concept consists in the solution of real problems in which the ratio of two quantities is used. The student should operate with markers, drawings, or work with symbols, depending upon his ability to interpret and deal intelligently with ratios in problems. The following problems illustrate the method to use.

1. If the cost of $\frac{3}{4}$ yard of ribbon is 12¢, what is the cost per yard?

In this problem, the student should recognize that the ratio of the cost (given) of $\frac{3}{4}$ yard to the cost (missing) of one yard would be $\frac{3}{4}$. He may put 3 disks on one pile and 4 disks on another pile to represent the ratio of the two numbers. He may continue adding 3 chips to one pile and 4 chips to the other pile until he has 12 chips in the smaller pile. Then he should see that there would be 16 chips in the other pile. Therefore, 16¢ is the cost when $\frac{3}{4}$ of the cost is 12¢. The students may make a diagram to show the ratio of the two amounts. The diagram on the left shows that three of its four parts represent 12¢, hence one part would represent 4¢ and four parts would represent the total cost, or 16¢.



It should be understood that here the ratio of the cost of a piece of ribbon to the cost of a yard is $\frac{3}{4}$. This statement can be expressed in the form, $\frac{3}{4} = \frac{12}{?}$. Now the student should think, "$3\overline{)12} = 4$," hence both terms of the fraction, $\frac{3}{4}$, must be multiplied by 4 to have the numerator of the equivalent fraction

to be formed equal to 12. Then the denominator of the given fraction would be multiplied by 4 to find the denominator of the equivalent fraction. The notation, $\frac{3}{4} = \frac{12}{?}$, is similar to the notation expressing a *proportion* A proportion shows the equality of two ratios, as $\frac{2}{3} = \frac{6}{9}$. If some letter, as x, were substituted for the question mark (?) in the equality, $\frac{3}{4} = \frac{12}{?}$, the resulting equation would be a proportion. The student must understand how to solve a fractional equation before he can solve an equation which represents a proportion.

The student who has advanced to the stage in which he is able to deal with symbols without the use of supplementary aids should reason in this manner: "If $\frac{3}{4}$ of the cost is 12¢, the cost will be $\frac{4}{3}$ as much as 12¢, or $\frac{4}{3} \times 12¢ = 16¢$" This solution is based on principle No. 5 given on page 144. This principle states that when the product of two numbers and one of the numbers are given, the missing number can be found by dividing the product by the given number.

The problem also can be solved as follows:

If the cost of $\frac{3}{4}$ yard $= 12¢$
The cost of $\frac{1}{4}$ yard $= 4¢$ ($\frac{1}{3}$ *of 12¢*)
Then the cost of $\frac{4}{4}$ yard $= 16¢$ (*4 × 4¢*)

This procedure may be designated as the method of *unitary analysis* as described on page 242. This method is easy for the student to understand. The use of the ratio concept is at a higher level of quantitative thinking than the method of unitary analysis.

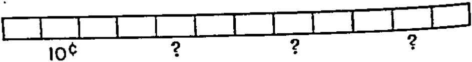
2. At the rate of 3 oranges for 10¢, find the cost of a dozen oranges.

The student should recognize that it is easy to solve the problem by finding the ratio of 3 oranges to 12 oranges, and by using this ratio to find the cost of a dozen oranges. The ratio of 3 oranges to 12 oranges is $\frac{1}{4}$. If $\frac{1}{4}$ dozen oranges cost 10¢, the cost per dozen would be $4 \times 10¢$, or 40¢. Following the same procedures used in the other problem, the student should utilize the kind of material to find a solution as determined by the level

of abstraction at which he is able to work. He should understand that when he multiplies 10¢ by 4, he multiplies by the reciprocal of the fraction $\frac{1}{4}$ which is the ratio of the two quantities.

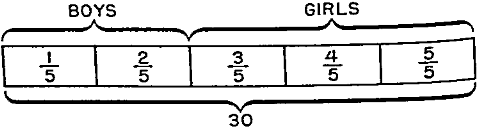
As illustrated above, the notation similar to a proportion may be used to represent the ratio of the cost of a given number of oranges to the cost of a dozen oranges. Then the example would be written in the form $\frac{3}{10} = \frac{12}{?}$. The missing number would be found by multiplying both terms of the fraction, $\frac{3}{10}$, by the quotient of 12 divided by 3, or 4.

The diagram below shows a visual representation of the problem.



3. The enrollment of a class is 30. If there are $\frac{2}{3}$ as many boys as girls, how many boys are there and how many girls are there?

The level of abstraction at which the student is able to solve this problem will depend upon his background. He should understand that the fraction $\frac{2}{3}$ is interpreted to mean that the ratio of the number of boys to the number of girls is $\frac{2}{3}$. Only the very superior student will understand how to solve this problem without a diagram because he must discover that the number of boys is $\frac{2}{5}$ of the enrollment and the number of girls is $\frac{3}{5}$ of the enrollment. A diagram of the type shown would enable most students to see that one part on the diagram represents $\frac{1}{5}$ of 30, or 6. Then the number of boys would be 12 and the number of girls would be 18.



In all problems of the type involving the ratio of two quantities, it is important for the student first to identify the ratio before proceeding further in the solution. The level of abstraction at which he is able to operate depends upon his understanding of relationships among quantities.

In many problems it is necessary to find the ratio of two numbers, as in each of the following problems:

a. A team won 3 of the 5 games it played. What was the ratio of the number of games won to the number of games played?
b. From a, what was the ratio of the number of games lost to the number of games won? What was the ratio of the number of games won to the number of games lost?
c. Jane is 12 years of age and her older sister is 16 years of age. Express the ratios of their ages.
d. At 45¢ per dozen, what part of a dozen oranges can be bought for 30¢?

The use of objective materials or diagrams helps many students to compare two numbers. The student should use these supplementary learning aids until he is able to deal solely with symbols. All problems in which it is necessary to find what per cent one number is of another number involve finding the ratio of two numbers. If per cent is involved, the ratio is expressed as hundredths and then as a per cent.

Using Proportion Based on Athletic Records

Many students in junior high school are interested in athletic records, such as batting averages, team standings, and earned run averages. Each of these records is expressed as a ratio. In many cases the ratio is expressed as a three-place decimal. The teacher should have those students who are especially interested in compiling data about certain star athletes make special reports to the class on some of their findings. Data of the following kind, which show the records of some of the greatest batters in baseball history, typify the use of ratios in reporting these baseball records.

Home Runs Per Times at Bat

Babe Ruth	1 per 12 ab's (at bat)
Ralph Kiner	1 " 13 "
Ted Williams	1 " 15 "
Jimmie Foxx	1 " 15 "

Runs Batted In Per Times at Bat

Babe Ruth	1 per 3.80 ab's
Ted Williams	1 " 4.00 "
Lou Gehrig	1 " 4.02 "
Jimmie Foxx	1 " 4.23 "

Walks Per Times at Bat

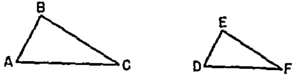
Ted Williams	1 per 4.86 ab's
Babe Ruth	1 " 5.09 "
Ralph Kiner	1 " 5.97 "
Lou Gehrig	1 " 6.30 "

From these data it is possible to predict approximately how many home runs, or runs batted in, any one of these stars should have had in a given number of times at bat. For example, on the average, how many home runs should Babe Ruth have hit in 100 times at bat? The ratio of the number of home runs to the number of times at bat was $\frac{1}{12}$. The number of home runs he should have hit in 100 times at bat would have been $\frac{1}{12}$ of 100, or approximately 8. On the average he should have hit approximately 8 home runs for each 100 times that he was at bat.

c. The Meaning and Use of Similar Figures

When Figures Are Similar

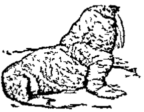
Ratios are used in work dealing with *similar* figures. Similar figures have the same shape. The triangles on page 351 have the same shape but different areas. If two figures have the same shape and size, they are both *congruent* (see page 354) and similar. Hence, congruent figures are similar, but similar figures need not be congruent.



A model is a miniature reproduction of some object, hence the model is similar to the object represented. If a model of a DC-7 airplane is made to *scale* $\frac{1}{150}$, this means that a part 1 inch long on the model is equal to 150 inches on the corresponding part of the airplane. The fraction, $\frac{1}{150}$, expresses the ratio of the length of a part on the model to the corresponding length of a part on the airplane.

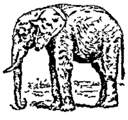
One of the most familiar uses of scale drawing is represented on maps. All maps are drawn to scale. The shape of the map is similar to the shape of the area represented. (The surface represented is assumed to be flat.) Most maps found in geographies and road maps are drawn to a scale in which 1 inch represents a certain number of miles. Many maps issued by the United States Geodetic Survey are drawn to a scale expressed as the fraction, $\frac{1}{63,360}$. The fraction, $\frac{1}{63,360}$, known as a *representative fraction*, shows that 1 inch on a map represents 63,360 inches, or 1 mile, on the earth's surface.

The teacher should have certain students find illustrations of representative fractions used in a dictionary and report the findings to the class. An assignment of this kind provides for differentiation of the curriculum in arithmetic for the superior students. A picture of an animal or a bird in a dictionary usually contains a representative fraction.



WALRUS
$\frac{1}{80}$

LEOPARD
$\frac{1}{30}$



AFRICAN ELEPHANT
$\frac{1}{125}$

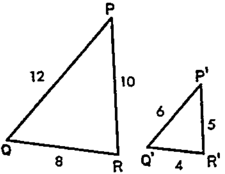
The scale of a drawing or picture may be expressed as a proper or an improper fraction or an integer. Thus, if the scale of a picture of an animal is 1, the picture is life size. A picture may be "blown up," which is the case in the illustrations of microscopic life or of insects. In dealing with these drawings the scale may be expressed by some number, as $\times 100$. This ratio indicates that the dimensions given in the picture are 100 times the dimensions of the true size of the object represented.

Corresponding Parts of Similar Figures

Two triangles are similar when the corresponding angles of the triangles are equal. In the two triangles shown, the *corresponding angles* are equal; therefore, the triangles are similar. Angles P and P′ are corresponding angles. The sides opposite these angles are *corresponding sides.* The sides QR and Q′R′ are corresponding sides and they are opposite angles P and P′, respectively.

In similar figures, the corresponding sides have equal ratios. In the triangles shown, the ratio of PR to P′R′ is $\frac{2}{1}$. Similarly, the ratio of the other two sets of corresponding sides is $\frac{2}{1}$. If two figures are similar, the corresponding angles are equal and the ratios of corresponding sides are equal. Hence the sides are *proportional.*

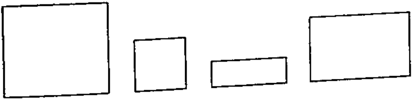
Two triangles are similar when their corresponding angles are equal. Two rectangles have equal angles, but these figures need not be similar. When the ratios of the corresponding sides are not equal, the rectangles are not similar.



In similar figures the corresponding angles are equal and the corresponding sides are proportional. If triangles have their corresponding angles equal, their corresponding sides must be proportional. This fact also is true concerning regular polygons. Therefore, to prove that two triangles or two regular polygons are similar, only the corresponding angles must be proven equal.

The Golden Section

The Greeks were interested in finding the ratio of the width to the length of a rectangle which would be most pleasing to the eye. Which of the following rectangles do you like best?



The Greeks decided that a rectangle is most pleasing to the eye when the ratio of width to length is the same as the segments of a line divided into "extreme and mean ratio." A line is divided into two segments in extreme and mean ratio when the ratio of the smaller of the two segments to the larger segment is equal to the ratio of the larger segment to the whole line. Thus, the longer of its two segments is equal to the square root of the product of the shorter segment and the entire line segment. If one line, a, is 5 inches and another line, b, is 8 inches, b is approximately the mean proportional between a and $a + b$ as 8 is equal approximately to the square root of the product of 5 and 13, or $\sqrt{65}$. This division of a line is known as the *golden section* and the ratio is sometimes called the *golden ratio*. Therefore, a rectangle having its dimensions in the ratio of approximately $\frac{5}{8}$ represented a rectangle which, the Greeks assumed, had the best proportions.

"The 'golden section' ratio is frequently found in the comparison of lengths that occur in nature, such as the distance from

the forehead to the nose and from the nose to the chin of a person, the distance from the bill of a bird to its legs and from its legs to the end of its tail, the distances between the joints of the fingers, etc."[1]

The teacher should have some of the students measure their classroom to find how the ratio of width to length compares with the golden ratio. Similarly, the students should compare the ratios of the sides of such rectangular objects as a rug having dimensions of $9' \times 12'$, or our national flag in which the ratio of width to length is 10:19, with the golden ratio.

The Terms Congruent, Similar, *and* Symmetric *Compared*

The teacher should have the student differentiate the various terms pertaining to the size, shape, and position of geometric figures. These terms about which the student has learned are *congruent*, *similar*, and *symmetric*, respectively.

Two identical gloves for the same hand are congruent. Two gloves having the same style but different sizes for the same hand are similar. Two gloves of a pair placed side by side are symmetric. The student should be able to give other illustrations to represent the difference between any two of these concepts.

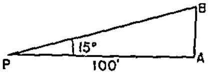
d. Methods of Applying Indirect Measurement

The Meaning of Direct and Indirect Measurement

If you measure the height of a flagpole by dropping a weighted tape from its top, you use *direct measurement*. If you find its height by measuring the length of its shadow or by measuring the angle its top intercepts from a given point, you use *indirect measurement*. In direct measurement, the measuring instrument is applied directly to the object measured. In indirect measurement, the measuring instrument is not applied to the object measured.

[1] Brandes, Louis G. "Recreational Mathematics for the Mathematics Classroom of Our Secondary Schools," *School Science and Mathematics* 54:620.

Stephen, Sister Marie "The Mysterious Number PHI," *The Mathematics Teacher*. 49:200-204.

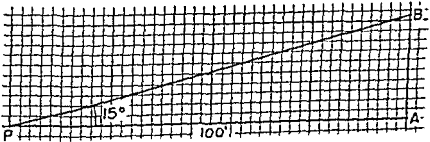


Indirect measurement is used in finding distances to inaccessible points, such as the distance from the earth to the moon.

By the completion of the junior high school, the student should know how to find the distance between two inaccessible points by at least the following four methods: (1) by using a scale drawing; (2) by measuring shadows (page 356); (3) by using proportions in similar triangles (page 358); (4) by applying the tangent ratio (page 360).

Finding Heights by Scale Drawing

It is possible to use scale drawing to find the height of a flagpole, AB, if the angle made by the line from the top of the pole with the horizontal is known, as angle P in the diagram. This angle is known as the *angle of elevation*. The student can make a *quadrant* or a *clinometer* to measure the angle at P. (On page 542, there is a description showing how to make a quadrant.) The point P should be selected at some distance easily measured and scaled, such as 100 feet, from the foot of the pole, or A. *If the angle at P is 15°, the height of the pole can be found by use of graph paper as shown. In the illustration,* 1" = 25'; therefore, PA *is 4 inches in length.* At the point P, draw an angle



of 15° and continue the side of this angle until it cuts the perpendicular from A at the point B. Then AB is the scaled distance equal to the height of the flagpole. The length of the line AB is 1.1 inches, therefore, the height of the pole is 1.1×25 feet, or approximately, 27 feet.

Students should be required to participate in field work which involves finding the height of inaccessible objects by indirect measurement. In this type of work the students should find the heights of such objects as trees, buildings, telephone poles, television aerials, or other objects near the school.

Finding Heights by Measuring Shadows

It is said that about the year 500 B.C., Thales, a famous Greek mathematician, measured the height of the Great Pyramid by comparing its height with the length of its shadow. When the

Students should have the opportunity to make indirect measurements in the field.

Public Schools, Chicago, Illinois

sun's rays make an angle of 45° with a vertical object and its shadow, the height of the object will be the same as the length of its shadow. In this case the ratio of the two dimensions is 1. Regardless of the ratio of the height of a vertical object and the length of its shadow, this ratio may be used in finding the height of an inaccessible object, if the length of the object's shadow is known.

If a stick 5 feet high casts a shadow 6 feet long, the ratio of the height of the stick to the length of its shadow is $\frac{5}{6}$. At the same time, the ratio of the height of any other nearby object and the length of its shadow would be $\frac{5}{6}$ because the triangles formed by the sun's rays with objects and their shadows would be similar. Since the length of a shadow of an object can usually be found by direct measurement, the ratio of the height of the given object to the length of its shadow can be used to find the height of the inaccessible object. The ratio of the height of the stick to the length of its shadow is $\frac{5}{6}$. Hence the height of a flagpole will be $\frac{5}{6}$ of the length of its shadow. If the length of the shadow is 60 feet, the height of the flagpole must be $\frac{5}{6} \times 60$, or 50. Therefore, the height of the flagpole is 50 feet.

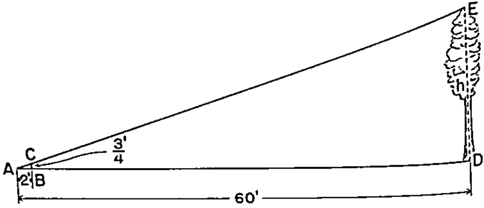
Frequently, a student does not know whether he should multiply the length of the shadow of an inaccessible object by the ratio of the height of a known object to the length of its shadow or by the reciprocal of this ratio. In the case cited above, the student would not be certain whether the multiplier should be $\frac{5}{6}$ or $\frac{6}{5}$. He should decide which ratio to use by considering whether the length of the shadow is greater or less than the height of an object. If the known height of an object is less than the length of its shadow, the unknown height of another object must be less than the length of its shadow. Then the length of the shadow must be multiplied by a ratio having a value less than 1 to find the unknown height. On the other hand, if the length of the shadow of an object of known height is less than the height of that object, then the length of the shadow of the object of unknown height must be multiplied by a ratio greater than 1 to find the unknown height.

The height of the flagpole can be found by writing the given data in the form of a proportion, as $\frac{5}{6} = \frac{?}{60}$, and then solving for

the missing number. The fraction equivalent in value to the ratio, $\frac{5}{6}$, is found by multiplying both terms of the fraction, $\frac{5}{6}$, by 10, the quotient of 60 divided by 6. If the student understands how to solve an equation, the example may be written as a proportion, as $\frac{5}{6} = \frac{h}{60}$. To solve this equation, multiply both of its members by 60 and then the value of h would be 50. Hence the height of the flagpole is 50 feet. Since a proportion is an equation, a missing term of a proportion may be found by solving the given equation. Thus, in the equation, $\frac{3}{5} = \frac{x}{25}$, x is equal to 15. It is apparent that a student should understand how to solve an equation before he attempts to solve a proportion by algebra. Of course a proportion may be solved by mechanical methods. In the proportion, $\frac{2}{3} = \frac{x}{15}$, the student can be shown how to find the value of x by cross multiplication, but he would not understand the process. After he understands how to solve the equation by multiplying both members by the same number, he may discover that cross multiplication will give the same result as multiplying both numbers by 15.

Finding Distances by Use of Similar Triangles

It is possible to find the distance to an inaccessible point by use of similar triangles. Triangles ABC and ADE are similar

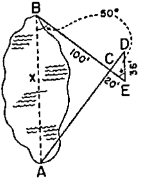


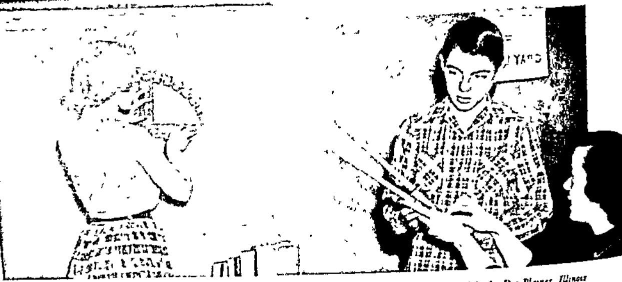
because the corresponding angles are equal. The height of the tree, h, is the distance between the points D and E. Since the sides of the triangles are proportional, $\frac{AB}{BC} = \frac{AD}{h}$. Substituting known values in the proportion, the equation becomes $\frac{2}{\frac{3}{4}} = \frac{60}{x}$, or $\frac{8}{3} = \frac{60}{x}$. Multiplying both members by $3x$, the equation becomes $8x = 180$, or $x = 22.5$. Therefore, the height of the tree is 22.5 feet.

To solve the equation, $\frac{2}{\frac{3}{4}} = \frac{60}{x}$, it is necessary to simplify the complex fraction, $\frac{2}{\frac{3}{4}}$. On pages 200–201 the reader saw how to simplify a complex fraction. This is a difficult procedure and it should be included only in a program for the superior students.

In the figure shown below, triangles ABC and CDE are similar because the corresponding angles are equal. The line AB represents the distance to an inaccessible point, A. Since the sides of the triangle are proportional, $\frac{BC}{AB} = \frac{CE}{DE}$, or $\frac{100}{x} = \frac{20}{36}$. Solving the equation, x is equal to 180. Therefore, the distance AB is 180 feet.

To find a missing dimension in a proportion formed by the ratios of two sets of corresponding sides of similar triangles, the student must know how to solve an equation. The teacher should





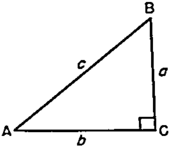
Public Schools, Des Plaines, Illinois

Students review measuring of angles before studying the tangent ratio.

be certain that the student has the necessary background to understand the work. Students in a typical eighth grade class rarely are adequately prepared to find a missing dimension of similar triangles by solving a proportion expressed as an equation. This type of work should be deferred until the ninth grade.

e. Finding Distances by Using the Tangent Ratio

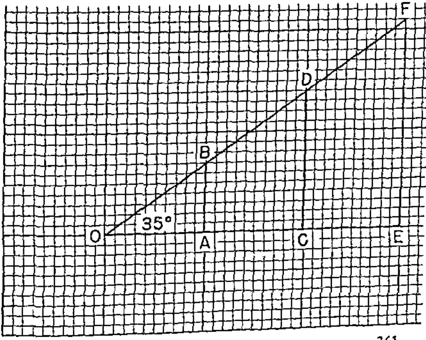
A fourth method for finding the distance to an inaccessible point is by the *tangent* (*tan*) *ratio*. The tangent ratio of an acute angle of a right triangle is the ratio of the arm opposite to the arm adjacent. In the triangle given, the sides are lettered in lower case to correspond to the opposite angles. The tangent of angle A is equal to the ratio of the side opposite, a, to the side adjacent, b, or $\tan A = \frac{a}{b}$. This ratio is always the same for an acute angle of a given number of degrees in any right triangle.

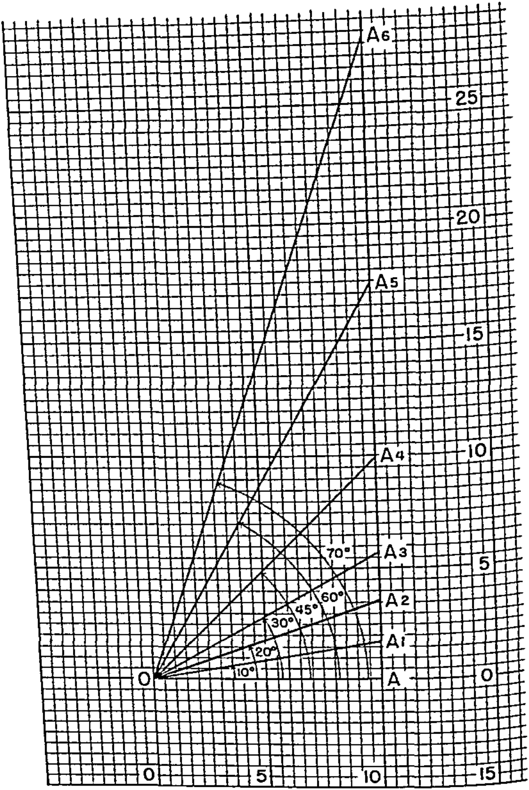


In the diagram below, triangles OAB, OCD, and OEF are similar. Angle O is common to each of these triangles and its value is 35°.

The tangent of angle O in triangle OAB is $\frac{AB}{OA}$. The tangent of this angle in each of the other two triangles is $\frac{CD}{OC}$ and $\frac{EF}{OE}$, respectively. If OA = 10, then the ratio of AB to OA is $\frac{7}{10}$, or .7. Similarly, the ratio of EF to OE is $\frac{21}{30}$, or .7. The data in the diagram show that the tangent of an angle of 35° is approximately .7.

Since the tangent ratio of a given acute angle in a right triangle is always the same for that angle, a knowledge of this ratio makes it possible to solve for the missing arm of a right *triangle when one of the arms is given.* In triangle OAB, if AB *represents the height of a structure when the angle of elevation at* O *is* 35°, the height *would* be .7 *times the* horizontal distance OA. The values of the tangent of acute angles *of different sizes* are helpful in solving for the missing arm of a *right triangle when* the value of one of the arms is given.





A6
25
20
A5
15
A4
10
70°
A3
5
60°
45°
A2
30°
A1
20°
10°
O
A
0
0
5
10
15

From the graph, it is possible to find the approximate values of angles of 10°, 20°, 30°, 45°, 60°, and 70°. The tangent of an angle of 10° is equal to the ratio of AA_1 to OA. The value of AA_1 is equal to approximately 1.8 and the value of OA is 10. Therefore, the ratio between these amounts is equal to $\frac{1.8}{10}$, or .18. From the graph, show that the ratios of the other angles given are equal approximately to .36, .57, 1, 1.7, and 2.7, respectively. If one of the acute angles of a right triangle is 45°, the other acute angle also is 45°; hence the arms must be equal and their ratio must be 1.

The teacher should have the students use graph paper on which they should draw acute angles of different sizes and then estimate the value of the tangent of each angle. Usually the tangents of angles less than 60° can be read to the nearest tenth from graphs drawn with care. From the drawings, the class should be able to discover the following properties of a tangent of an angle:

1. As the size of the angle increases, the value of the tangent increases.
2. The tangent of an angle less than 45° is less than 1.
3. The tangent of an angle of 45° is 1.
4. The tangent of an acute angle greater than 45° is greater than 1.
5. It is not possible to find the tangent of an angle of 90°.
6. Doubling an angle does not double its tangent.

A Table of Tangents

Since the tangent of an acute angle is always the same, tables of values are available which give the tangent of each acute angle. The table on page 364 gives the tangents of the angles from 1° to 89° inclusive.

The use of the table can be shown by solving the following problem:

At a distance of 1500 feet from a searchlight throwing a vertical beam on a cloud, the angle of elevation of the light spot on the cloud was 40°. What was the height of the cloud?

Table XI. Tangent Ratios

Angle	Tangent	Angle	Tangent	Angle	Tangent
1°	0.017	31°	0.601	61°	1.804
2	0.035	32	0.625	62	1.881
3	0.052	33	0.649	63	1.963
4	0.070	34	0.675	64	2.050
5	0.087	35	0.700	65	2.145
6	0.105	36	0.727	66	2.246
7	0.123	37	0.754	67	2.356
8	0.141	38	0.781	68	2.475
9	0.158	39	0.810	69	2.605
10	0.176	40	0.839	70	2.747
11	0.194	41	0.869	71	2.904
12	0.213	42	0.900	72	3.078
13	0.231	43	0.933	73	3.271
14	0.249	44	0.966	74	3.487
15	0.268	45	1.000	75	3.732
16	0.287	46	1.035	76	4.011
17	0.306	47	1.072	77	4.331
18	0.325	48	1.111	78	4.705
19	0.344	49	1.150	79	5.145
20	0.364	50	1.192	80	5.671
21	0.384	51	1.235	81	6.314
22	0.404	52	1.280	82	7.115
23	0.425	53	1.327	83	8.144
24	0.445	54	1.376	84	9.514
25	0.466	55	1.428	85	11.430
26	0.488	56	1.483	86	14.301
27	0.509	57	1.540	87	19.081
28	0.532	58	1.600	88	28.636
29	0.554	59	1.664	89	57.290
30	0.577	60	1.732		

If h represents the height of the cloud, then $\frac{h}{1500} = \tan 40°$.

From the table, the tangent of 40° is 0.839, hence $\frac{h}{1500} = .839$.

Multiplying both numbers by 1500, the equation becomes

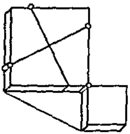
$$h = 1500 \times .839, \text{ or } 1258.5.$$

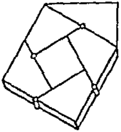
The approximate height of the cloud was 1260 feet.

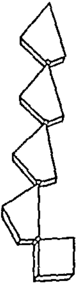
f. Square Root and Its Application

The Right Triangle

A square constructed on the *hypotenuse* of a right triangle has the same area as the sum of the squares constructed on the two arms of the triangle. This principle was known to the ancient Greeks and is known as the Pythagorean Theorem. A hinged device may be made out of wood to illustrate this principle.

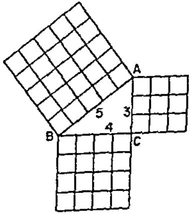






In the figure on the right, $5^2 = 3^2 + 4^2$. A triangle of this kind is known as a 3-4-5 triangle and is a right triangle.

The surveyors of ancient Egypt made use of the 3-4-5 principle to lay out a right triangle. They put knots on a rope at equal intervals and stretched the rope over pegs placed in the ground so that 3 knots, 4 knots, and 5 knots, respectively, would be on the sides of the triangle formed. This triangle would be a right triangle.



The Meaning of Square Root

The *square of a number* is equal to the product of a number multiplied by itself. Therefore, the *square root* of a number is one of its two equal factors. Thus, the square of 6 is 36 and the square root of 36 is 6.

N	N^2	N	N^2
1	1	10	100
2	4	20	400
3	9	30	900
4	16	40	1600
5	25	50	2500
6	36	60	3600
7	49	70	4900
8	64	80	6400
9	81	90	8100

The student should make a table similar to the one shown on the right. The table should prove the following things pertaining to the square root of perfect squares:

1. The square root of a one- or two-place whole number is a one-place number.

2. The square root of a three- or four-place whole number is a two-place number.

3. From the above generalizations, the student should discover that the square root of a five- or six-place whole number is a three-place number.

By using the above generalizations and referring to the table, the student should see that the square root of a whole number, such as 375, must be a two-place number. This root must be between 10 and 20. Similarly, the square root of 7256 must be a two-place number between 80 and 90.

In a right triangle with c as the hypotenuse, $c^2 = a^2 + b^2$. If any two of the three sides are known, it is possible to solve the equation to find the third side of the triangle. The values of c, a, and b in the above equation are as follows:

$$c = \sqrt{a^2 + b^2}$$

$$a = \sqrt{c^2 - b^2}$$

$$b = \sqrt{c^2 - a^2}$$

To find the missing side of a right triangle when two sides are given, it is necessary to know how to find the square root of a number.

Ways of Finding Square Root

There are many different ways of finding the square root of a number. Some of the most familiar ways are:

1. By use of a table (see page 368)
2. By approximation
3. By division
4. By the algorism employing a new trial divisor for each figure of the root
5. By use of a slide rule or a computing machine.

There are few social applications of square root which most students will meet. For that reason, the student should not learn to find square root by a method which will soon be forgotten, as the method of determining a new trial divisor for each figure of the root as shown in the algorism on the right.

	$\sqrt{1\ 2'9\ 6}$	3 6
	9	
60	3 9 6	
+6	3 9 6	
66		

A method which all students can learn consists in finding the square root of a number from a table of squares and square roots. The table on page 368 gives the squares and square roots of the first 150 whole numbers.

A number which has an exact square root is a *perfect square*, as the number 144. An approximate square root of a number which is not a perfect square can be found from the table. Thus, the square root of 210 is between 14 and 15. The square of 14 is 196 and the square of 15 is 225. Since 210 is about midway between 196 and 225, the approximate square root of 210 is 14.5. The student should understand the meaning of an approximate square root before using a table of the type shown.

The teacher can enrich the work dealing with square root for the superior students by having them learn to find the approximate square root of a number by division. To find the square root of 476 by this method, the student should first make a good estimate of the root. He should see that the root will be a two-place number. The root must be between $\sqrt{400}$ and $\sqrt{900}$. Since $20^2 = 400$ and $30^2 = 900$, the root must be much nearer

Table XII. Squares and Square Roots

N	N^2	$\sqrt{N}$	N	N^2	$\sqrt{N}$	N	N^2	$\sqrt{N}$
1	1	1.00	51	2,601	7.14	101	10,201	10.05
2	4	1.41	52	2,704	7.21	102	10,404	10.10
3	9	1.73	53	2,809	7.28	103	10,609	10.15
4	16	2.00	54	2,916	7.35	104	10,816	10.20
5	**25**	**2.24**	**55**	**3,025**	**7.42**	**105**	**11,025**	**10.25**
6	36	2.45	56	3,136	7.48	106	11,236	10.30
7	49	2.65	57	3,249	7.55	107	11,449	10.34
8	64	2.83	58	3,364	7.62	108	11,664	10.39
9	81	3.00	59	3,481	7.68	109	11,881	10.44
10	**100**	**3.16**	**60**	**3,600**	**7.75**	**110**	**12,100**	**10.49**
11	121	3.32	61	3,721	7.81	111	12,321	10.54
12	144	3.46	62	3,844	7.87	112	12,544	10.58
13	169	3.61	63	3,969	7.94	113	12,769	10.63
14	196	3.74	64	4,096	8.00	114	12,996	10.68
15	**225**	**3.87**	**65**	**4,225**	**8.06**	**115**	**13,225**	**10.72**
16	256	4.00	66	4,356	8.12	116	13,456	10.77
17	289	4.12	67	4,489	8.19	117	13,689	10.82
18	324	4.24	68	4,624	8.25	118	13,924	10.86
19	361	4.36	69	4,761	8.31	119	14,161	10.91
20	**400**	**4.47**	**70**	**4,900**	**8.37**	**120**	**14,400**	**10.95**
21	441	4.58	71	5,041	8.43	121	14,641	11.00
22	484	4.69	72	5,184	8.49	122	14,884	11.05
23	529	4.80	73	5,329	8.54	123	15,129	11.09
24	576	4.90	74	5,476	8.60	124	15,376	11.14
25	**625**	**5.00**	**75**	**5,625**	**8.66**	**125**	**15,625**	**11.18**
26	676	5.10	76	5,776	8.72	126	15,876	11.23
27	729	5.20	77	5,929	8.78	127	16,129	11.27
28	784	5.29	78	6,084	8.83	128	16,384	11.31
29	841	5.39	79	6,241	8.89	129	16,641	11.36
30	**900**	**5.48**	**80**	**6,400**	**8.94**	**130**	**16,900**	**11.40**
31	961	5.57	81	6,561	9.00	131	17,161	11.45
32	1,024	5.66	82	6,724	9.06	132	17,424	11.49
33	1,089	5.74	83	6,889	9.11	133	17,689	11.53
34	1,156	5.83	84	7,056	9.17	134	17,956	11.58
35	**1,225**	**5.92**	**85**	**7,225**	**9.22**	**135**	**18,225**	**11.62**
36	1,296	6.00	86	7,396	9.27	136	18,496	11.66
37	1,369	6.08	87	7,569	9.33	137	18,769	11.71
38	1,444	6.16	88	7,744	9.38	138	19,044	11.75
39	1,521	6.24	89	7,921	9.43	139	19,321	11.79
40	**1,600**	**6.33**	**90**	**8,100**	**9.49**	**140**	**19,600**	**11.83**
41	1,681	6.40	91	8,281	9.54	141	19,881	11.87
42	1,764	6.48	92	8,464	9.59	142	20,164	11.92
43	1,849	6.56	93	8,649	9.64	143	20,449	11.96
44	1,936	6.63	94	8,836	9 70	144	20,736	12.00
45	**2,025**	**6.71**	**95**	**9,025**	**9.75**	**145**	**21,025**	**12.04**
46	2,116	6.78	96	9,216	9.80	146	21,316	12.08
47	2,209	6.86	97	9,409	9.85	147	21,609	12.12
48	2,304	6.93	98	9,604	9.90	148	21,904	12.17
49	2,401	7.00	99	9,801	9.95	149	22,201	12.21
50	2,500	7.07	100	10,000	10.00	150	22,500	12.25

to 20 than to 30. These students also should learn a quick means to square a two-place number ending in 5. (See page 524 for the method to use.) The square of 25 is 625, therefore, the square root of 476 is between 20 and 25. Since 476 is nearer to 400 than to 625, the student should estimate some number, as 22, to be the square root. Then he should divide the given number by the estimated root and express the quotient to the nearest tenth as shown on the right. If the exact root is estimated, the quotient will be the same as the divisor. If divisor and quotient are different, as in the illustration, the average of these two numbers usually will be a close approximation of the square root of the dividend. The average of 22 and 21 6 is equal to $\frac{22.0 + 21.6}{2} = \frac{43.6}{2}$, or 21.8. The product of 21.8 and 21.8 is 475.26. Therefore, the approximate square root of 476, to the nearest 0.1, is 21.8.

```
      21.6
  22)476.0
     44
      36
      22
      140
      132
```

Sometimes it may be necessary to find the approximate square root of a five- or six-place whole number. For example, the distance between bases on a baseball diamond is 90 feet. From this fact it is possible to find the distance from home plate to second base. This distance is equal to the square root of $90^2 + 90^2$, or 16,200. Two factors of 16,200 are 100 and 162. The square root of 16,200 is equal to the product of the square root of each of its two factors, 100 and 162. The square root of 100 is 10 and the square root of 162 is a little less than 13. If 162 is divided by 13, the approximate quotient will be 12.5. The average of 13 and 12.5 is 12.75. Therefore, the approximate square root of 16,200 will be 10×12.75, or 127.5. The distance on a baseball diamond from home plate to second base is approximately 127.5 feet.

```
      12.5
  13)162.0
     13
      32
      26
       60
```

To find the approximate square root of a five- or six-place whole number, round off the number to the nearest hundred. Next, express the rounded number as the product of two factors, one of which is 100. Then find the square root of each of these factors. The product of these two roots will be the approximate

square root of the given number. To find the square root of 235,475, proceed as follows:

1. Round off 235,475 to the nearest hundred, as 235,500.
2. Express 235,500 as 100 × 2355.
3. The square root of 100 is 10; the approximate square root of 2355 is 48.5.
4. The approximate square root of 235,475 is equal to 10 × 48.5, or 485.

Square Root of Decimals

Further enrichment can be provided for the fast learners by having them find the square root of pure decimals, such as .1. Many college students will give .1 or .01 as the square root of .1. The students should understand that the square root of a pure decimal also will be a pure decimal because 1 squared is 1, hence the square root of a number less than 1 also must be less than 1. Then it is well for the teacher to review with the class at this time some of the principles learned in multiplication of decimals.

The following two principles should be understood:

1. The product of tenths and tenths is hundredths.
2. The product of hundredths and hundredths is ten thousandths.

From these two principles it is seen that it is not possible to find the square root of a pure decimal expressed as tenths. The number .1 must be expressed or regrouped as .10. Then it is possible to find the approximate square root of .10. The approximate root would be .3 or .32. A decimal, such as .625, should be expressed as ten thousandths, or as .6250. Then the approximate square root would be .79 and not .25 which frequently is given as the square root of .625.

Many students at the high school or college level are surprised to find that the square root of a common fraction or a decimal fraction is larger than the fraction itself. The product of the fractions is less than either factor. Hence the root of a fraction must be greater than the fraction itself.

Interesting Property of Perfect Squares

The teacher should show an interesting property of perfect squares. A perfect square is equal to the sum of as many consecutive odd numbers beginning with 1 as there are in the root of the number. This fact can be shown by the sequence of numbers which follow:

Odd Numbers:	1	3	5	7	9	11	13
Squares:	1	4	9	16	25	36	49

The sum of the first six consecutive odd numbers $(1 + 3 + 5 + 7 + 9 + 11)$ is 36, a perfect square. In the same way each of the squares in the above sequence can be found.

The fact that a perfect square is equal to the sum of as many consecutive odd numbers as there are in the root of the number makes it possible to find the square root of the number by use of a calculating machine. After a perfect square is recorded in the machine, the consecutive odd numbers, beginning at 1, are subtracted until there is not a remainder. The number of subtractions is the square root of the number.

Questions, Problems, and Topics for Discussion

1. Give a problem in which it is necessary to compare two numbers. Compare these numbers in as many different ways as possible.

2. Give a diagrammatic solution of the following problems:
   a. The cost of construction of a road $\frac{3}{8}$ mile long was \$45,750. At that rate, find the cost to construct a mile of road.
   b. A recipe for making cherry preserves calls for $1\frac{1}{2}$ pounds of sugar to 2 pounds of cherries. Find the number of pounds of each ingredient needed to make 10 pounds of preserves.

3. The representative fraction of a map is $\frac{1}{10,000}$ Find the distance between two places which are 75 centimeters apart on the map Express the distance between the two places in kilometers.

4. In which of the following rectangles is the ratio of width to length closest to the golden ratio: a. 12′ × 15′? b. 9″ × 15″? c. 32″ × 45″? d. 10 cm. × 15 cm.?

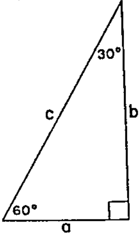
5. Assume that the corresponding angles in each set of the figures named are equal. Select the sets of figures which are similar: a. Two squares b. Two acute triangles c. Two hexagons d. Two regular octagons e. Two parallelograms f. Two trapezoids.

6. State in which of the following the missing dimension must be found by indirect measurement: a. The diameter of the moon b. The height of a structure c. The weight of a battleship d. The height of a mountain peak e. The depth of a lake.

7. A yardstick in a vertical position casts a shadow 45 inches long and at the same time a tower casts a shadow 280 feet long. What is the height of the tower? [Ans. 224 ft.]

8. Solve problem 7 by using proportion.

9. At a distance of 250 feet from the base of a tower, the angle of elevation of the top of the tower is 35°. Find the height of the tower. Solve by scale drawing and check the answer by use of the tangent ratio. [Ans. 175 ft.]



10. In the figure above, if $c = 2$ and $a = 1$, find side b. [Ans. $\sqrt{3}$]

11. The triangle shown is known as a 30-60-90 triangle. In a triangle of this kind, the arm forming the 60° angle is always half the hypotenuse. Find the arms of a 30-60-90 triangle having a hypotenuse of 8 inches. [Ans. 4 in.; $4\sqrt{3}$ in.]

12. In the above triangle, find the numerical value of the tangent of 30°; of 60°. Compare your answers with the values of these angles as given in the table on page 364. [Ans. tan 30° = .58; tan 60° = $\sqrt{3}$]

13. The *cotangent* (cot) of an angle is the ratio of the side adjacent to the side opposite. Find the cotangent of 30°; of 60°. How does the cotangent of an angle compare with the tangent of that angle? [Ans. cot 30° = $\sqrt{3}$; cot 60° = .58]

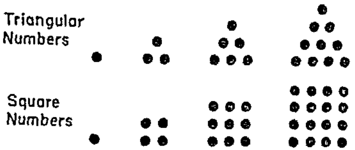
14. What is the value of an angle in which the tangent and the cotangent would be the same?

15. How does a 30-60-90 triangle differ from a 3-4-5 triangle?

16. The altitude of an equilateral triangle bisects the base. Find the area of an equilateral triangle having a side of 8 inches. [Ans. 27.2 sq. in.]

17. Find the approximate square root of a. 125, 475 b. 83, 416 c. 260, 187 d. 534, 876.

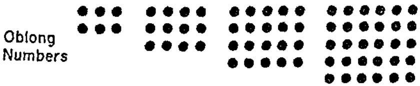
18. Use division to find the approximate square root of 240. [Ans. 15.5]



19. The arrangement of the dots in the diagram above represents "triangular numbers." *Beginning with 1, add each triangular number with the previous number.* The figure found will be a "square number." Make drawings similar to the above to represent the first seven triangular and square numbers.

20. *Prove that the sums of the consecutive odd numbers beginning with 1 are* equal, respectively, to the number of dots in each square formed in problem 19.

21. Write the first six consecutive even numbers, beginning with 2. Add the *first two numbers, then the first three numbers, and so on, in order. Prove* that these sums may be represented by dots arranged in oblongs to form a pattern similar to that shown below.



Suggested Readings

Buckingham, B. R. *Elementary Arithmetic, Its Meaning and Practice*, pp. 580–617. Boston: Ginn and Co., 1947.

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Reeve, William D. *Mathematics for the Secondary Schools*, pp. 301–330. New York: Henry Holt and Co , 1954.

Shuster, Carl N. and Bedford, Fred L. *Field Work in Mathematics*, pp. 47–58. New York: American Book Co., 1935.

Chapter 11

Informal Geometry

THIS chapter deals with the teaching of the following topics:

a. How to measure lines and angles

b. The basic geometric constructions

c. How to find perimeters and areas of the most familiar plane figures

d. How to find the volume of the most familiar solids

e. The meaning of congruence and symmetry.

a. How to Measure Lines and Angles

The Scope of Geometry

Students in junior high school mathematics classes learn about *informal* or *intuitive* geometry. Students in senior high school mathematics classes learn about *formal* or *demonstrative* geometry. Intuitive geometry deals predominantly with the size and shape of figures while demonstrative geometry deals largely with the proof of theorems. In informal geometry the proof of the equality of two figures is established by measurement as contrasted with the procedure in demonstrative geometry when a formal and logical proof is required. The following discussion pertains to informal or intuitive geometry.

The elements of geometry are *points*, *lines*, *surfaces*, and *solids*. A *geometric figure* is any combination of these elements. A moving point forms a *line*, for example, the line formed by a moving

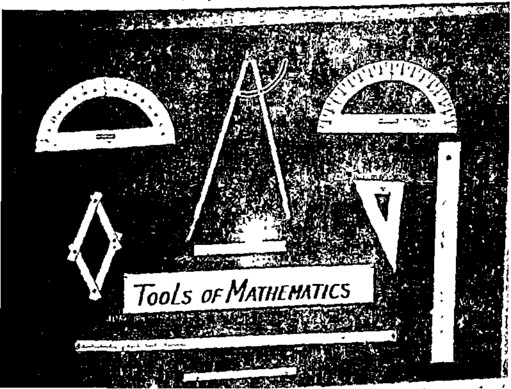
pencil point. A moving line forms a *surface*, for instance, a paint brush moved on a wall. A moving surface forms a *solid*, for instance, the path made by a snow plow in a bank of snow.

A discussion of the teaching of informal geometry should be concerned with the study of the *size*, *shape*, and *position* of figures. A *closed figure* is bounded by lines or surfaces. The size of a closed figure is expressed in terms of its area or volume. The shape of a closed figure is identified as a triangle, rectangle, or some other plane figure, or as some kind of solid. Two surfaces having equal areas and the same shape are *congruent* These figures may be placed in such a position that they will be *symmetric* as is discussed on page 354.

Instruments for Work in Geometry

A student should have a ruler or straightedge (a ruler without graduations), compasses, and a protractor for work in informal geometry. The classroom should be equipped with blackboard compasses, a blackboard protractor, a T-square, a drawing board, and a drawing triangle.

Photo by Ted Bartels, Photo Club, Ridgewood High School, New Jersey

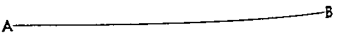


In most of the work in informal geometry, a student either *draws* or *constructs* angles and figures. For example, to draw a figure, a student would use a ruler and a protractor. To find the midpoint of a line by drawing, he would measure from either extremity of the line a distance equal to half the known length of that line. To find the midpoint of a line by construction, he would use only a straightedge and compasses.

Measuring Lines

When we speak of a line AB, we refer to a *segment* of a line. A line has no fixed length, but a segment of a line represents that portion of a line which is between two given points on the line.

A student should learn to measure the lengths of given line segments. In the illustration, the length of AB is



slightly more than $2\frac{1}{2}$ inches. If the ruler used in measuring this line has the inch graduated into quarter inches, the length would be expressed as $2\frac{3}{4}$ inches; if graduated to eighths, the length would be expressed as $2\frac{5}{8}$ inches; if graduated to sixteenths, the length would be expressed as $2\frac{11}{16}$ inches. Illustrations of this kind should prove to the student that *no measurement is ever exact but is always approximate.* The precision of a measurement depends upon the measuring instrument and the skill of the person making the measurement. In the illustration above, if the smallest division given on the ruler were the unit indicated, the length of the line would be expressed as follows:

To the nearest inch,	3 inches
To the nearest half inch,	$2\frac{1}{2}$ inches
To the nearest quarter inch,	$2\frac{3}{4}$ inches
To the nearest eighth inch,	$2\frac{5}{8}$ inches
To the nearest sixteenth inch,	$2\frac{11}{16}$ inches

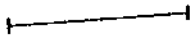
Very probably all students at the junior high school level would be able to measure the length of the line to the nearest inch, half inch, and quarter inch. It is likely that the measurements of the length of the line made by different students would

vary when measured to the nearest eighth or sixteenth of an inch. In this case error would result from the incorrect use of the measuring instrument. Errors in this case would be due to:

1. Inability to read the graduations of the inch
2. Inability to determine the extremities of the line
3. Failure to have the initial point of the line at the zero point on the ruler
4. Inability to determine the assigned value when the terminal point of the line does not fall on a scaled point of the ruler.

The teacher should instruct the student how to reduce these four types of errors to a minimum. For example, if a student is unable to read a ruler on which the inch is graduated to eighths or sixteenths, an enlarged model or drawing on the blackboard of an inch graduated to these units should enable the student to discover how to read the graduations. A demonstration ruler, on which the inch is enlarged, is an effective instructional aid for teaching how to measure with a foot ruler. The teacher can point to different graduations of the inch and have the student indicate the values represented. Then the learner should be able to find the corresponding values on his own ruler.

The student can guard against the second type of error by marking the extremities of a line as shown on the right. He should understand why a sharp pencil should be used in marking the terminal points.

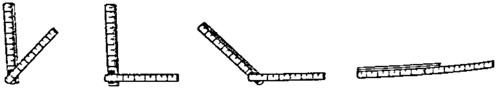


The third type of error can be reduced by having the student place the beginning point of the line at the 1-inch mark or at some other inch mark, but not at the zero point on the ruler. This is especially true when the zero point on the scale coincides with one end of the ruler.

The value to be assigned a point between two scaled values on a ruler is determined in the same way that a value is assigned when a number is rounded off. If the point is nearer one scaled value than the other, the point is assigned the scale value of the point to which it is nearer. If the point is midway between two scaled values, the point is given the value of the greater of the two scaled values.

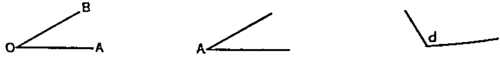
Measuring Angles

If a line is rotated in the same plane about a point, an *angle* is formed. The size of the angle depends upon the amount of rotation from the original position of the line to its final position. The *vertex* is the point common to both sides. It is easy to demonstrate how an angle is formed by opening one section of a carpenter's rule. As a section of the rule is opened, the rotation of the section about a fixed point on the rule forms or generates an angle. The most familiar kinds of angles are:



Acute Right Obtuse Straight

An angle may be designated by three different notations as shown. A number may be used instead of a lower case letter to represent an angle, as the number 2 instead of any letter, as *d*.



The instrument used for measuring an angle is a *protractor*. The student should discover why a protractor contains both an inner and an outer scale in order to make it easy to use in measuring angles. On one scale, the zero point is on the left and on the other scale, this point is on the right. The center of the semicircle is marked on the protractor by a "crowfoot." This center point always should be placed at the vertex of the angle to be measured. The zero point on the scale should be on one side of the angle. Then the point on that scale cut by the other side of the angle would indicate the number of degrees in the angle. The student should check his measurement by deciding if the angle is acute or obtuse. With the exception of those angles having a value of approximately 90°, it is easy to see whether an angle is acute or obtuse.

If the sides of an angle to be measured are not long enough to cut the arc of the protractor, a straightedge, such as a piece of paper, should be placed along one side of the angle so that the zero point on the protractor can be identified on this side of the angle. When the zero point on the protractor has been identified, the straightedge may be removed and placed on the other side of the angle to find the point of intersection of the side and the arc of the protractor. An exercise of this kind should prove to the student that the size of an angle is not affected by the length of its sides.

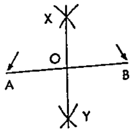
b. The Basic Geometric Constructions

The Constructions Appearing in Designs and in Nature

Many of the beautiful designs and patterns found in architecture and in nature can be reproduced by use of compasses and straightedge. In order to reproduce these motifs, the student must know certain basic methods of performing fundamental constructions, such as:

1. How to construct a perpendicular bisector of a line
2. How to bisect an angle
3. How to construct a perpendicular to a line from a given point not in the line
4. How to construct a perpendicular to a line at a given point in the line
5. How to construct an angle equal to a given angle.

Constructing a Perpendicular Bisector of a Line



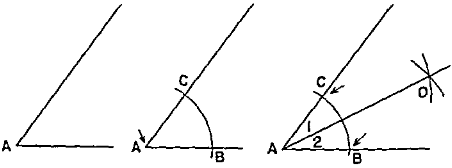
The figure shows how to construct the perpendicular bisector of a given line segment. The line XY is the perpendicular bisector of AB, hence two right angles are formed. By use of a protractor the student should measure each angle. The student must also understand that two points determine a line. In all

constructions it is necessary to have two given points in order to draw a line between them. Frequently, a student has one given point on his construction and guesses the second point to which to draw the line, such as a bisector of an angle or a line.

The teacher should have the students understand why the radius used for describing the intersecting arcs must be greater than half AB. He should understand what would happen if the radius used were half of AB.

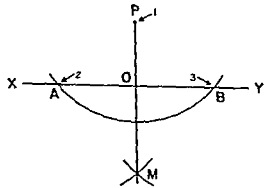
Bisecting an Angle

The figures shown below show the steps in bisecting an angle.



The student should bisect an angle and then check the accuracy of the construction by measuring with a protractor to find out if the two angles formed are equal.

Constructing a Perpendicular to a Line

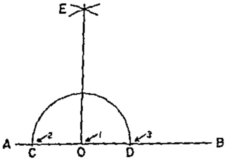


The figure on the left shows how to construct a perpendicular to a line XY from a given point P not in the line. The teacher must be alert to see that the student locates the point M correctly. P is the given point. In order to have

a perpendicular to XY, there must be a second point, as M. Many students guess the midpoint of the line segment AB and connect this point with P. The point M may be on the same side of the line XY as the point P or the point M may be on the opposite side of XY as shown in the figure. The class should discuss how the accuracy of the construction is affected by the distance between the point M and the given point P

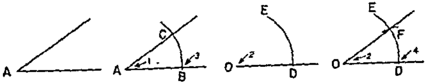
Constructing a Perpendicular from O on AB

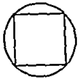
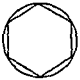
The figure on the right shows how to construct a perpendicular to a line from any given point on the line, as the point O on AB. The reader should be able to follow the sequence of steps in the construction. The numbers indicate where the point of the compasses should be placed.



Constructing Angle O Equal to Angle A

The figures below show how to construct an angle equal to a given angle, as the angle A. Since the size of the angle depends upon the amount of rotation about a point, as the point A, describing an arc, BC, with any convenient radius, as AB gives a distance BC to be used as a radius to form an intersecting arc at F. The points O and F determine the side OF of the angle O. Angle O is equal to angle A.



Inscribed Polygons

The student should be able to construct certain *regular polygons*, such as an *inscribed square*, an *inscribed octagon*, and a *regular hexagon.* A regular polygon is a plane figure which has equal sides and equal angles. A polygon is inscribed in a circle when the vertexes of the polygon are on the circle, as shown in the drawings above. To construct an inscribed square, the student should draw a diameter in a given circle and then construct the perpendicular bisector of this diameter. If the four points at the extremities of the diameters are joined in order, the figure formed will be an inscribed square. It is easy to find the points for the vertexes of an inscribed octagon by constructing the perpendicular bisector of each of the four sides of an inscribed square and continuing the bisectors until they cut the circle. Then join the consecutive points.

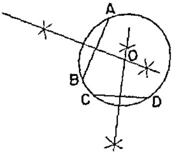
The first step in constructing a regular hexagon is to mark a point on a circle. From this point as a center and a radius equal to the radius of the circle, mark an intersecting arc on the circle. From this point of intersection as a center and with the same radius, describe another intersecting arc on the circle. Continue the process until an intersecting arc is finally made at the starting point on the circle. Then connect consecutive points on the circle by straight lines. The figure will be a regular hexagon. If alternate points on the circle are joined, the figure will be an *equilateral* triangle.

An equilateral triangle having sides of any given length can be constructed by using each end of the given side as a center with a radius equal to the given side and describing intersecting arcs. Then connect this point with the extremities of the given side. An equilateral triangle may be constructed either by this method or by inscribing a triangle in a circle as previously described.

In order to make fractional cut-outs, the student should find the points on the circumference by construction rather than by measurement. These points should be established so as to enable the student to divide circles into thirds, fourths, sixths, and eighths. Chapter 6 shows how the student should use the cut-outs in dealing with fractions.

Other applications of construction are the constructing of the medians and altitudes of a triangle. A median of a triangle connects a vertex with the midpoint of the side opposite. The medians of a triangle intersect at a common point. In a similar manner, the altitudes of a triangle meet in a common point. The altitudes of an obtuse triangle meet outside of the triangle. These constructions must be made with care to show that either the medians or the altitudes meet in a common point. Work of this kind can be given as enrichment for the students who prove to be superior in this phase of arithmetic as shown in Chapter 14. Sometimes the students who are poor in computation or problem solving are superior in construction work.

It is possible to find the center of a circle by construction. The center of a circle formed by marking around the base of a cylindrical can or a drinking glass can be found by drawing any two non-parallel *chords* and then constructing perpendicular bisectors of them. A chord is a straight line connecting two points on the circle. In the figure shown below, AB and CD are chords. The perpendicular bisectors of these chords intersect at O which is the center of the circle.



Copying Designs

Many students derive pleasure from copying designs or making original designs of their own. Often a study of snow crystals, rock crystals, or shape of flowers will enable a student to make interesting designs. The designs in the photograph illustrate the kinds that students at the junior high school level should be able to copy or enlarge. Many other designs are possible, using only straightedge and compasses. The student must be able to construct certain basic figures in order to make good reproductions.

Boys and girls enjoy using their compasses freely. Often the same basic design, with various shadings for the separate parts, can have a very different appearance. Later the students learn to apply the basic constructions to their designs, and they make designs based on hexagons and equilateral triangles, squares and octagons, as well as circles.

Public Schools, Montclair, New Jersey

c. How to Find Perimeters and Areas of the Most Familiar Plane Figures

The Formula for the Perimeter of a Square

A *formula* is a statement in mathematical language of a general rule or principle. Frequently, the first letter of a word is used to represent that value in a formula. The following formula can be written in the shorthand of mathematical language as illustrated:

The perimeter of a square	is equal to	four times	the length of a side
p	$=$	$4 \times$	
or	$p = 4s$		

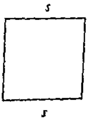
It is not necessary to write the symbol for multiplication in a term in a formula, as $4s$ or lw The term, $4s$, means 4 times s. Similarly lw means $l \times w$.

Many students find it difficult to distinguish between *perimeter* and *area*. A demonstration with a carpenter's rule or a square cut from cardboard should help to clarify the difference between these two concepts. The student should see that the distance around the square is its perimeter and the surface of the figure is its area. A linear unit is used to measure perimeter. A square unit is used to measure surface, or area.

The formula for the perimeter of a square may be written as follows:

$$p = 1s + 1s + 1s + 1s = 4s$$

or $p = 4s$

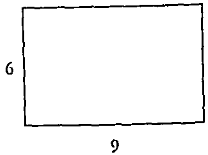


A student often is unable to understand how the formula, $p = 4s$, is derived from the expression, $p = s + s + s + s$. The numerical *coefficient* means the number which tells how *many times the quantity is taken, as $\underline{1}s$.* When the coefficient, 1, is not expressed, the coefficient indicates 1 of that quantity. This number should be written until the student discovers that the value of s means $1s$.

In a similar manner, the students should be able to work out the formula for the perimeter of an equilateral triangle.

The Formula for the Perimeter of a Rectangle

The class should discover different ways of finding the perimeter of a rectangle. In the rectangle on the right the students should find the perimeter in the following ways:



1. $6 + 9 + 6 + 9 = 30$

2. $\begin{array}{r} 6 \\ +6 \\ \hline 12 \end{array} \quad \begin{array}{r} 9 \\ +9 \\ \hline 18 \end{array} \qquad\qquad \begin{array}{r} 12 \\ +18 \\ \hline 30 \end{array}$

3. $\begin{array}{r} 6 \\ \times 2 \\ \hline 12 \end{array} \quad \begin{array}{r} 9 \\ \times 2 \\ \hline 18 \end{array} \qquad\qquad \begin{array}{r} 12 \\ +18 \\ \hline 30 \end{array}$

4. $\begin{array}{r} 6 \\ +9 \\ \hline 15 \end{array} \qquad\qquad\qquad \begin{array}{r} 15 \\ \times 2 \\ \hline 30 \end{array}$

The class should discuss which method is shortest. Very probably the students would decide that the fourth way is shortest. Then this way should be expressed as a formula. The formula may be written as

$$p = 2\ (l + w)$$

The teacher should capitalize on this opportunity to discuss the mathematical significance of the expression, $2\ (l + w)$. On page 143 one of the mathematical principles governing multiplication states that to multiply an indicated sum by a given number, each term must be multiplied by that number. Therefore, the student should discover that both the length and the width must be multiplied by 2. The values represented by l and w may be added first and then multiplied by 2, or each term may be multiplied by 2 and then the products added. The four solutions given above may be used to verify the mathematical principle formulated.

Area of a Rectangle

A student should not find areas of rectangles until he understands the meaning of area and the unit of measurement of area. Area is the amount of space that a surface contains or covers. *The unit of measurement of area is a square.*

Each student should have a 1-inch square cardboard to measure the area of a given rectangle. Of course the area of this square is 1 square inch. The student should draw a rectangle having given dimensions, for example, 3 inches by 5 inches. Then he should use his 1-inch square and mark the number of square inches that are in one row in the figure. This experiment should prove to him that there will be 5 square inches in each row and in 3 rows there will be 15 square inches in the area of the rectangle. By repeating this process in rectangles of different dimensions, the student should discover that the number of squares in a figure will be the same as the product of the number of linear units in the length and width. At this point he should attempt to formulate both the rule and the formula for the area of a rectangle. The rule for the area of a rectangle may be stated as follows: The area of a rectangle is equal to the product of length and width when both are expressed in the same linear unit. The formula is

$$A = lw$$

The steps in the derivation of the rule and the formula for the area of a rectangle may be summarized as follows:

1. Use a square unit, such as a square inch or a square foot, to find the number of squares needed to cover the surface of a given rectangle.

2. The student should discover that the number of square units in a row multiplied by the number of rows will be equal to the number of square units in the rectangle. Thus, the area of a 3 inch by 5 inch rectangle is equal to 3 times 5 sq. in., or 15 sq. in.

3. The student should discover that the product of the number of linear units in length and width will be the number of square units in any rectangle.

4. The student should write the rule for finding the number of square units in any rectangle as a formula. When written as a formula, the rule reads $A = lw$.

After the student understands the formula, he should use this shortcut method of expressing a rule for finding the area of any rectangle. Thus, the computation for finding the area of a rectangle 16 feet long and 9 feet wide would be as shown. The answer should be labeled properly to indicate the unit of measure used to express the area. It should be clear that the teacher should not have the student memorize a meaningless and erroneous rule of the type, "feet times feet gives square feet."

$$\begin{array}{ll} A = lw & \\ l = 16 \text{ ft.} & \\ w = 9 \text{ ft.} & 16 \\ A = ? & \underline{\times 9} \\ & 144 \\ \text{Area} = 144 \text{ sq. ft.} & \end{array}$$

The teacher should lead the students to understand why a square is the unit of measurement for area. Each student should draw several rectangles of any equal size, as 3 × 4 in., 2 × 6 in., etc. Then he should attempt to fill the space in each rectangle with squares and also with other figures, such as equilateral triangles, circles, or hexagons. The class should discuss the results of their experiments. The experiments should show why a square is considered the most acceptable unit for expressing area. The difficulties of using squares to find the area of a $3 \times 4\frac{1}{2}$ inch rectangle also should be shown.

For enrichment, the experiment could be continued starting with figures other than rectangles to be filled. It will be found that equilateral triangles, for instance, will fill equilateral triangles, but not any other figure.

The Formula for the Area of a Square

A square is a particular rectangle in which length and width are equal. Therefore, l may be substituted for w or w for l in the formula, $A = lw$. Then the formula would become $A = w \times w$ or w^2, or $A = l \times l$ or l^2. The letter s generally is used to represent the side of a square, hence the formula for the area becomes $A = s \times s$ or s^2. The small ^{2} is an *exponent* and it shows how many times s is used as a factor.

Maximum Area; Minimum Perimeter

At the junior high school level, one exercise for enriching the experience of the superior student consists in finding the kind of rectangle which will have the greatest area with the smallest perimeter. The student should find the areas of different rectangles which can be enclosed with a given amount of fencing, as 100 feet. The table on the right shows different values for w, l, and A when the perimeter is 100. The student should discover that a square will enclose a greater area than any other rectangle when the perimeter of the figures is constant. The proof of this statement is easily verified by the use of the calculus.

Width	Length	Area
25	25	625
20	30	600
15	35	525
10	40	400
5	45	225
1	49	49

The student whose imagination is stimulated by the data in the table should observe the relationship as the width continues to decrease. He would discover that the area also decreases. When the width becomes zero, the area of the rectangle becomes zero. In this case, there would be a straight line which would have a length of 50 feet. The student should discover that the area of a rectangle depends upon the dimensions. The dependence of one quantity upon another quantity represents the *function concept* as discussed in Chapter 12.

Properties of Parallelograms

When the student is first introduced to each kind of plane and solid figure, he should be encouraged to point out illustrations of that type of figure either in the classroom or in places outside of the classroom, preferably in both places. One of the objectives of teaching informal geometry should be to make the student conscious of the shape of figures in the world about him. The teacher should have the student identify such figures as parallelograms, triangles, or prisms before he finds their areas or volumes.

The teacher should have a student demonstrate a rectangle with a carpenter's rule. Then the student should apply pressure

at one corner so as to change the shape of the figure. The figure formed would be a *parallelogram* which is not a rectangle. The students should make the following discoveries about a parallelogram:

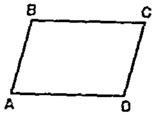
1. The opposite sides are equal
2. The angles may or may not be right angles
3. By measurement the student should prove that the opposite angles are equal
4. Figures in the shape of parallelograms are not rigid.

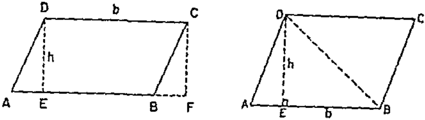
From the discoveries made, the students should formulate a rule for characterizing a parallelogram. The rule may be stated as follows:

A parallelogram is a plane figure having four sides in which the opposite sides are equal.

The opposite sides are seen to be parallel. From this fact the figure derives its name. Then the students should discover that a rectangle can be classified as a parallelogram, but a parallelogram need not be a rectangle.

The student has learned that each angle of a rectangle is 90°, hence the sum of the angles of a rectangle is 360°. Similarly, the sum of the angles of a parallelogram is 360°. The student has discovered that opposite angles of a non-rectangular parallelogram are equal. Therefore, the two *adjacent* angles must be unequal. If C represents an opposite angle to angle A, and B represents an adjacent angle, $2A + 2B = 360°$, or $A + B = 180°$. If angles A and B are not right angles, one of the angles must be less than 90° and the other angle must be more than 90°. Hence one angle is an acute angle and the other angle is an obtuse angle. In the parallelogram shown, angles A and C are acute and angles B and D are obtuse.





The Formula for Finding the Area of a Parallelogram

Either of two ways may be used to derive the formula for the area of a parallelogram, depending upon the background of the student. If he has learned how to find the area of a rectangle but not the area of a triangle, then he should know that the formula for finding the area of a parallelogram can be based on the formula for finding the area of a rectangle. On the other hand, if he understands how to find the area of a triangle, he should be able to derive the formula for finding the area of a parallelogram by dividing a parallelogram into two triangles.

The student should cut a parallelogram, as ABCD, from paper. Then he should cut off the right triangle ADE and place it as shown, forming the rectangle EFCD. The area of the rectangle is DE $\times$ EF in which DE is equal to the altitude of the parallelogram and EF is equal to the base, AB, of the parallelogram. Therefore, *the area of a parallelogram is equal to the product of the base and the altitude.* The formula for the area is:

$$A = bh$$

If the student knows that the formula for the area of a triangle is $A = \frac{1}{2}bh$, he can use this fact to derive the formula for the area of a parallelogram. The student should cut a parallelogram along a diagonal BD as shown in the drawing. By placing one triangle on the other triangle so that the corresponding sides match, he would discover that the two triangles have equal areas. The altitude of the triangle ABD is the same as the altitude of the parallelogram and the two figures have the same base. The formula for the area of the triangle is $A = \frac{1}{2}bh$, therefore, the formula for the area of the parallelogram would be:

$$A = 2 \times \tfrac{1}{2}bh, \text{ or } A = bh$$

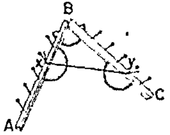
Perimeters Constant; Areas Variable

The teacher can use a carpenter's rule to show how to transform a rectangle into a parallelogram. The superior student should discover that a parallelogram formed by changing the shape of a rectangle has the same perimeter as the rectangle. On the other hand, the areas of the two figures are not the same. The length of the rectangle and the base of the parallelogram do not change, but the altitude of the parallelogram drawn from the base becomes shorter as the acute angles of the parallelogram become smaller. If the base remains unchanged but the altitude becomes shorter, the area of the parallelogram becomes smaller. When the acute angles of the parallelogram approach zero, the altitude would approach zero, and the final figure would be transformed into a straight line.

Properties of Triangles

The student has learned that neither a rectangle nor a parallelogram is rigid. He should perform an experiment to discover whether or not a triangle is rigid, by using three splints or strips of wood. He should bore a hole about a half inch from the end of each strip and fasten the corners with a screw or some other means, so as to form a triangle. If the corners are properly secured, the student should discover that it is not possible to change the shape of the triangle without breaking one of its sides. A carpenter's rule also may be used to demonstrate the same property of a triangle. A demonstration with materials of the type described or other similar materials should prove that a triangle is rigid. Because they are rigid, triangular figures are found frequently in construction work and in other places in which a high degree of rigidity is necessary.

The class should cut triangles from paper and measure with a protractor to find the sum of the angles in each triangle. The results of the measurements should prove that the sum is 180°. This same fact can be demonstrated by tearing off two of the corners of a triangle and placing them on either side of the third angle.



The diagram shows a very effective teaching aid to enable a student to discover how many degrees there are in the sum of the angles of a triangle. AB and BC are two strips of wood about two feet long. They are fastened so as to pivot at B. A protractor attached at B is used for measuring the angle at this point. Each arm of the angle is grooved so that a protractor may be fastened in this groove. Four or five nails, spaced along each arm, are partially driven in so that a string, such as XY, can be stretched from a nail on one arm to a nail on the other arm. In this way different triangles may be formed. The student should place a protractor in the groove by the nail used as a vertex of a triangle and read the number of degrees in each angle and also the number of degrees in the angle at B. Then he should find the sum of the three angles. The experiment should be repeated several times with different triangles. The results should prove that the sum of the angles of a triangle is equal to 180°.

Size of Angles in Triangles

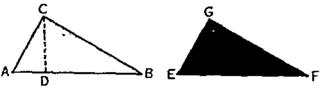
To provide enrichment for superior students, the teacher should have the students make certain discoveries about the size of angles in a triangle. These students should discover the answers to such questions as:

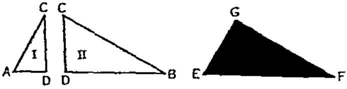
The student should assign numerical values to the different kinds of angles to prove his answers. He may measure angles of triangles of various shapes also. Then he should formulate the following generalizations:

1. A triangle cannot have one angle equal to a straight angle.
2. A triangle may have one and only one right angle or one obtuse angle, but not both.
3. A triangle may have three acute angles.
4. Every triangle must have at least two acute angles.
5. A triangle may have all three of its angles equal. Then each angle must be an acute angle.
6. A right triangle may have only two of its angles equal.
7. A triangle may have all three of its angles unequal.

In a similar manner, the superior student should discover how the sides of a triangle may vary. He should make drawings to prove that a triangle may have three equal sides, two equal sides, or all of its sides unequal.

The superior student should be able to draw both a right triangle and an obtuse triangle which has two of its sides equal.



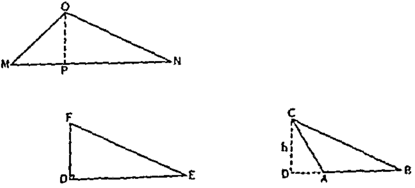


Developing the Formula for Finding the Area of a Triangle

If the student has learned how to find the area of a parallelogram, he can derive the formula for the area of a triangle from the formula for the area of a parallelogram. If a parallelogram *is cut along a diagonal, two equal triangles are formed.* Since the formula for the area of a parallelogram is $A = bh$, the formula for the area of a triangle would be $A = \frac{1}{2} bh$.

The formula for the area of a triangle also can be derived from the formula for the area of a rectangle If one of two triangles of the same size and shape is cut along an altitude and the two parts of the triangle are placed along the sides of the other triangle as on page 394, a rectangle is formed. Each student in the class should cut two equal triangles from construction paper and perform the experiment. This experiment will prove that the rectangle formed has the same base as the triangle and the width of the rectangle is the same as the altitude of the triangle. Since the area of the rectangle is equal to the area of two equal triangles, the area of a triangle is equal to half the area of the rectangle.

It is important for the student to understand that the position of the altitude in a triangle depends upon the kind of triangle. The diagrams below show the different positions which the altitude may occupy. The teacher should have models of obtuse, right, and acute angle triangles and have the students identify the position of altitudes in each of the triangles.



Evaluating the Formula for the Area of a Triangle

The teacher should have the student evaluate the formula for the area of a triangle and discover that the order of the multiplication of the factors does not affect the value of the product. He should decide which method is easier: whether to multiply either factor by $\frac{1}{2}$ (or divide the factor by 2) or to multiply the factors and then divide that product by 2.

A. $A = \frac{1}{2} \times 8 \times 15$

B. $A = \frac{1}{2} \times 9 \times 16$

C. $A = \frac{1}{2} \times 7 \times 9$

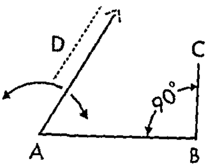
The student should discover from the above illustrations that the computation will be easiest in A if 8 is divided by 2 and that quotient multiplied by 15. In B, 16 should be divided by 2 and that quotient multiplied by 9. In C, the product of 7 and 9 should be divided by 2. From similar illustrations, most students should be able to make the following generalizations about the area of a triangle:

1. The area is equal to the product of the altitude and half the base.
2. The area is equal to the product of the base and half the altitude.
3. The area is equal to half the product of the base and the altitude.

The teacher also should emphasize the basic principle of multiplication given on page 144 which states that to multiply an indicated product by a number, only one factor is multiplied by that number. In the indicated product, $A = \frac{1}{2} \times 8 \times 12$, either 8 or 12 may be multiplied by $\frac{1}{2}$, but not both of these factors, without changing the value of A.

Properties of Trapezoids

A *trapezoid* is a closed plane figure of four sides which has two and only two parallel sides. A student should cut a triangle from



paper. By holding the scissors parallel to the base and cutting off the top of the triangle, a trapezoid is formed.

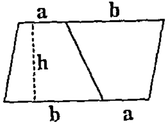
The figure above shows a piece of equipment which can be used to stimulate the imaginations of superior students. BC is a wooden stick which is stationary and perpendicular to AB. The stick AD may rotate at the point A. If the upright pieces form right angles and another bar or stick connects these pieces, a certain kind of figure is formed. The teacher should have the students indicate what kind of figure this is. If AD and BC are equal, the figure formed would be a rectangle; if they are unequal, a trapezoid would be formed. If AD is moved from a vertical position so as to make angle BAD acute, a trapezoid is formed, providing CD is parallel to AB. If AD intersects BC, a triangle is formed.

Following a demonstration with the frame, the superior student should make the following generalizations:

1. A trapezoid may have two right angles providing these angles are adjacent angles and not opposite angles.
2. It is not possible to have only one right angle in a trapezoid.
3. The two parallel sides of a trapezoid can never be equal.

From a study of other trapezoids, the following additional generalizations should be made:

4. If two of the angles of a trapezoid are acute, the other two angles must be obtuse.
5. All four angles of a trapezoid may be unequal.
6. The non-parallel sides of a trapezoid may be either equal or unequal.



The Formula for Finding the Area of a Trapezoid

The student should cut two equal trapezoids from paper and letter each upper base a and each lower base b. Then he should arrange the trapezoids as shown in the drawing. The resulting figure is a parallelogram. The base of the parallelogram is seen to be equal to the sum of the upper and lower bases of each trapezoid. The altitude of the parallelogram is the same as the altitude of the trapezoid. Therefore, the area of the trapezoid is half the area of the parallelogram. The formula for the area of a trapezoid is:

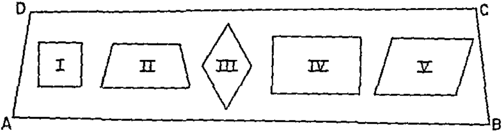
$$A = \tfrac{1}{2}h\,(a + b)$$

in which a and b are the bases

The formula for the area of a trapezoid illustrates the principle governing multiplication of an indicated product by a given number. Either of the two factors, h and $(a + b)$, is to be multiplied by $\frac{1}{2}$ or divided by 2. Since the order of multiplication does not affect the product, the rule for finding the area of a trapezoid may be stated in three different ways:

1. The area is equal to the product of half the altitude and the sum of the bases.
2. The area is equal to the product of the altitude and half the sum of the bases.
3. The area is equal to half the product of the altitude and the sum of the bases.

All students in the class should be able to substitute numbers in the formula for the area of a trapezoid to prove that each rule is correct. Only the superior student would be able to give the mathematical reason why each rule is valid.



Quadrilaterals

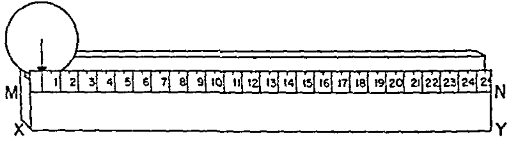
A *quadrilateral* is a closed plane figure of four sides. Students at the junior high school level should be familiar with squares, rectangles, parallelograms, and trapezoids. The superior students can help prepare a poster showing these and other kinds of quadrilaterals and then give the characteristics of each figure. The poster should be cut so that all of its sides and angles are unequal, as illustrated. The picture shows five special kinds of quadrilaterals and an irregular kind. Very probably most of the class would be unfamiliar with a *rhombus*. A rhombus is a quadrilateral which has four equal sides, but the adjacent angles may or may not be equal. If the adjacent angles are equal, the figure is a square. The students should discover that the diagonals of a rhombus bisect each other at right angles.

Finding the Circumference of a Circle

Each student, or each small group of students, should use a cylindrical can or a wheel to find the ratio of the circumference

Boys and girls find π experimentally.

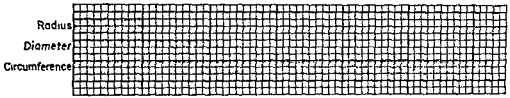
Public Schools, Washington, D. C.



to the diameter. The class should compare the ratios of different size cans or wheels. The value of each ratio should be a little more than 3. Then the teacher should have a class demonstration with a 7-inch wooden disk to show how to find the ratio of the circumference to the diameter. A student should mark a point on the circumference and then roll the disk along a yardstick until the marked point touches the stick. The scaled value of this point should be approximately 22 inches. This distance divided by the diameter would give a quotient of $\frac{22}{7}$, or $3\frac{1}{7}$, which may be expressed approximately as 3.14.

The diagram shows an effective instructional aid for demonstrating the method of finding the ratio of the circumference to the diameter. In the diagram, XY is a board about 24 inches long and MN is part of a yardstick or a cloth tape fastened to the side of the board. At the base of XY there is a groove about $\frac{3}{8}$ inch wide in which circular disks of plywood about $\frac{1}{4}$ inch thick can be rolled. If the teacher has disks of 3″, 4″, and 7″ in diameter, a student can give a demonstration to show the circumference of each circular disk. Then the class can find the ratio of the circumference to the diameter for each disk. The experiment should prove that the circumference is about 3.1 times the diameter.

The teacher should have the students compare the results of their experiments with the ratio $3\frac{1}{7}$. Most of the results should be approximately 3.1. The ratio of the circumference to the diameter is π and its value is approximately $3\frac{1}{7}$, or 3.14. The exact value of π can not be determined, but in beginning work with this symbol, π is usually given the value of $3\frac{1}{7}$, or 3.14. The more precise value, 3.1416, is used when greater accuracy is demanded than that used at the junior high school level.



The experiments should prove that the circumference of any circle is equal to π times its diameter, or π times twice the radius. The formula for the circumference of a circle is

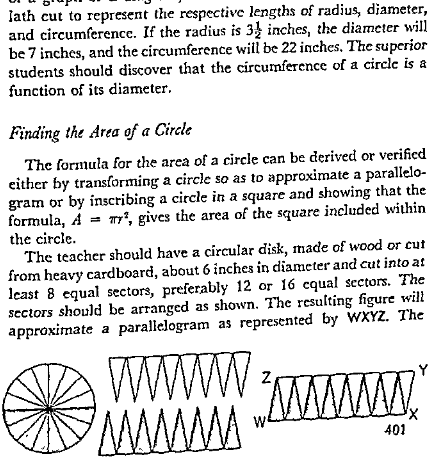
$$C = \pi d \quad \text{or} \quad C = 2\pi r$$

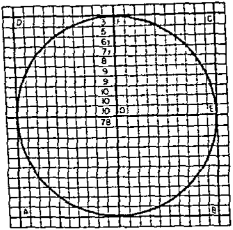
The teacher should have the students make a graph, similar to the graph shown, to display on the bulletin board. Instead of a graph or a diagram, it would be better to have pieces of lath cut to represent the respective lengths of radius, diameter, and circumference. If the radius is $3\frac{1}{2}$ inches, the diameter will be 7 inches, and the circumference will be 22 inches. The superior students should discover that the circumference of a circle is a function of its diameter.

Finding the Area of a Circle

The formula for the area of a circle can be derived or verified either by transforming a circle so as to approximate a parallelogram or by inscribing a circle in a square and showing that the formula, $A = \pi r^2$, gives the area of the square included within the circle.

The teacher should have a circular disk, made of wood or cut from heavy cardboard, about 6 inches in diameter and cut into at least 8 equal sectors, preferably 12 or 16 equal sectors. The sectors should be arranged as shown. The resulting figure will approximate a parallelogram as represented by WXYZ. The

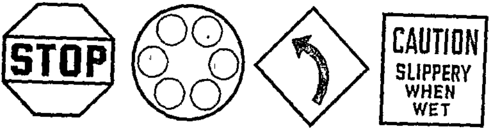




greater the number of equal sectors into which the disk is cut, the closer will the figure formed by these sectors approximate a parallelogram. The altitude of the parallelogram is equal to the radius of the circle. The base of the parallelogram is approximately half of the circumference, or $\frac{2\pi r}{2}$, or πr. Substituting r for h and πr for b in the formula, $A = bh$, the formula for the area of a circle is shown to be:

$$A = \pi r \times r, \quad \text{or} \quad A = \pi r^2$$

According to the second method of deriving the formula for the area of a circle, the student should draw on heavy cardboard, cross ruled in inches, a square having a side of 20 inches. Then he should draw a circle with a radius of 10 inches and use the center of the square as the center of the circle. Next, he should draw two radii at right angles as shown. The sector FOE is a quarter of the circle and the square FOEC has an area of 100 small squares. The area of the sector FOE may be found by counting the number of small squares which are entirely within the circle and adding that number to the number of squares in which half or more of a square is within the circle. The number at the end of each row of squares indicates the number of squares which are considered to be within the circle for that row. The sum of these numbers is 78. Therefore, 78 of the 100 squares in each quarter of the large square are within the circle. Hence the number of squares in the whole circle must be 4×78 or 312.



The formula for the area of a circle is $A = \pi r^2$. If 10 is substituted for r, the formula becomes 100π, or 314. The area found by counting the number of squares is 312. The experiment should show that approximately the same result would be found by the graphic method and by the formula, $A = \pi r^2$.

Neither of the methods of proof is easy to understand, nor is the proof satisfactory from a mathematical point of view. The students who have developed a keen insight into number should be able to understand the procedures described. It is doubtful if the other students will understand the derivation of the formula for the area of a circle.

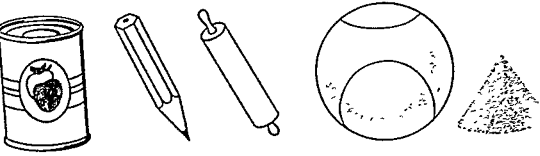
The Geometry of Road Signs

By the time the student has learned to find the circumference of a circle, he has dealt with triangles, quadrilaterals, pentagons, hexagons, and octagons. A fine unit dealing with the shape of plane figures consists in exploring with the students the geometry of highway signs. The students should report on the shapes of different signs seen along highways and the kind of information each sign contains. The illustrations above show some of the kinds of signs used along highways and the kind of information each sign conveys.

d. How to Find the Volume of the Most Familiar Solids

Developing the Formula for Finding the Volume of Prisms

A *solid* is a figure having three dimensions. The *volume* of a solid is the amount of space it encloses. Just as a square is the unit of measurement of the surface of a plane figure, a *cube* is the unit of measurement of the volume of a solid.

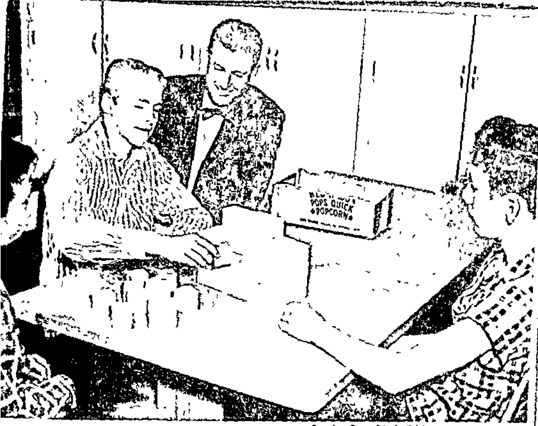


The picture above shows some of the more familiar solids.

Most students at the junior high school level are familiar with ice cubes. A student knows that the number of ice cubes in a tray of cubes can be found by multiplying the number of cubes in one row by the number of rows. This product will be the number of cubes in one tray. The number in one tray multiplied by the number of trays (of equal size) is equal to the number of cubes in all of the trays of a refrigerator. If the trays are stacked on top of each other, the cubes may be considered as arranged in layers. Thus, if a tray contains 8 cubes in a row and there are 2 rows, the tray will contain 16 cubes. If there are 3 trays or layers of cubes, there will be 48 ice cubes in the refrigerator.

The student should understand that even if each of two refrigerators manufactures 48 ice cubes, the capacities of the freezing units of the two machines need not be equal, because the capacity of each freezing unit depends upon the size of the cubes. In order to compare the freezing capacity of the two refrigerators, it is necessary to express the volume of each in the same standard unit. Instead of an ice cube which has different sizes, a cube having a standard size, such as a cubic inch or a cubic centimeter, should be used. A cube having an edge expressed in standard linear units is the cubic unit for measuring volume.

The teacher should have approximately 125 1-inch cubes and rectangular boxes of different dimensions. The dimensions of certain cigar boxes are 8″, 5″, and 2″. The students should fill a box of this size with cubes. He should discover that 8 of these cubes can be placed in one row and there will be 5 rows, or a total of 40 cubes in one layer. Since there are 2 layers, the box holds 80 cubes. He should fill enough boxes of different size to



Stanislaus County Schools, California

When the student handles cubes, he learns the meaning of volume and discovers that "product of the dimensions" is a shortcut.

enable him to discover that the product of the number of units of length and width will give the number of cubic units in one layer. This product multiplied by the number of linear units in the height will be the number of cubic units in the volume of the box. When he has made this discovery, he has learned how to find the volume of any rectangular solid, or *right prism*. All prisms have two parallel bases that have the same shape and equal areas. The shape of the base determines the kind of prism. If the base of a prism is a triangle, the solid is a *triangular prism*.

The formula for the volume of a rectangular prism is:

$$V = lwh$$

The teacher should use this formula to illustrate multiplication principle No. 1 on page 143. This principle states that the order in which numbers are multiplied does not affect the product.

The student should use 1-inch cubes to discover the rule for finding the number of cubic inches a small cube will contain. A 2-inch cube built with these cubes will contain 8 cubes, a 3-inch cube will contain 27 cubes, and so on. After he builds a few such cubes, he should discover the rule for finding the number of cubic inches in any cube which is from 1 to 12 inches on a side. When he knows the rule, it should not be necessary for a student to arrange 1728 1-inch cubes as a cubic foot in order to find the number of cubic inches in a cubic foot. He should use the blocks as exploratory material to help him discover a basic principle. The teacher must ask the kinds of questions which will direct the student's thinking so that he will discover the relationship between the number of inches on each edge of a cube and the number of cubic inches in the volume of the solid.

In the formula $V = lwh$, lw gives the number of square inches in the base. B may be substituted for lw and then the formula for the volume of a prism is:

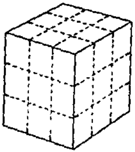
$$V = Bh$$

Now it is possible to find the volumes of prisms whose bases have different shapes, providing the areas of the bases and the heights are known.

Models of a Cubic Foot and a Cubic Yard

In order to have a student form a correct concept of the size of a cubic foot, the teacher should have a 1-foot cube made of wood or plastic. Each of its faces should be ruled into square inches. A square 1-foot board one inch thick also should be provided. The relative size of a square inch, a square foot, and a square yard can be shown on a chart.

The student should understand that if a piece one inch thick were cut from the 1-foot cube, the resulting piece would be the same as the square 1-foot slab. When this slab is ruled into cubic inches, it is easy to see that there would be 144 cubic inches in the block. Since there would be 12 such slabs in the cubic foot, there would be 12 × 144 cubic inches, or 1728 cubic inches in a cubic foot.



Many students do not understand the meaning of a cubic yard. The teacher should have a group of the students construct a frame that is a cubic yard in size and lay off on each edge two 1-foot lengths. These points should be connected by a string as shown. The diagram shows that there are 9 square feet in a square yard and 9 cubic feet in one layer of the cubic yard. Since there are three layers, there must be 27 cubic feet in a cubic yard. The model also shows the relative size of a cubic yard and a cubic foot. The model should enable a student to form a good concept of the size of a cubic yard of earth. This is the unit used to measure the amount of earth taken from an excavation. A cubic yard of earth is frequently known as a *load* of earth.

Anderson[1] showed how a model of a cubic yard may be so constructed that it would be collapsible. A model of this kind may be stored in a small space compared with the space needed for a rigid cubic model having an edge of a yard.

Formulas for Finding the Area and Volume of Cylinders

It is easy for the student to derive the formula for the *lateral* (side) *area* of a cylinder. He should remove the label of a cylindrical fruit can by cutting along an edge perpendicular to both bases. The label will form a rectangle in which the width is equal to the height of the can and the length is equal to the circumference of the base of the can. Substituting πd for l and

[1] Anderson, Sigfrid E. "Devices for a Mathematics Laboratory," *The Mathematics Teacher*, 46:578–579.

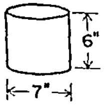
h for w in the formula $A = lw$, the resulting formula for the lateral area of a right cylinder is:

$$A = \pi dh \qquad \text{or} \qquad A = 2\pi rh$$

The student has learned that the formula for the volume of a prism is $V = Bh$. Regardless of the shape of the prism, this formula will apply in finding its volume. It follows, then, that the base of the prism may be either a polygon or a circle. Since the area of a circle is πr^2, this value may be substituted for B in $V = Bh$. Then the formula for the volume of a cylinder is:

$$V = \pi r^2 h$$

The teacher should have samples of the cans discussed on page 109. Both cans have the same height, but the diameter of one can is twice the diameter of the other can. The teacher should hold the cans so that the bottoms show and have the students estimate the relative diameters. In almost all cases a class will state that the smaller diameter is a little more than half the larger diameter. This illusion of size shows that the student's judgment is affected by the relative size of the areas of the two bases. After measuring to prove that the diameters are in the ratio of 1 to 2, the student should use measures of water to compare the volumes of the two containers. He will discover that the larger can will hold four times as much as the smaller can. The superior student should be able to discover this relationship by using the formula to find the volume. In the formula $V = \pi r^2 h$, π is a constant and h is the same, therefore, r is the quantity which varies. Since r is doubled, the volume would be multiplied by 2^2 or 4. The teacher should have had the superior student discover the change which takes place in the area of a square when the length of each side is multiplied by any given number, such as doubling each side.



The teacher can have the class make a cylindrical container that will hold a gallon. A cylindrical container having the dimensions of the cylinder shown in the drawing on the left will have a volume of 231 cubic inches. Substituting the

values given in the drawing and $\frac{22}{7}$ for π in the formula for the volume of a cylinder, the solution becomes:

$$V = \frac{22}{7} \times \left(\frac{7}{2}\right)^2 \times 6$$

$$V = \frac{\overset{11}{\cancel{22}} \times \overset{1}{\cancel{7}} \times 7 \times \overset{3}{\cancel{6}}}{\underset{1}{\cancel{7}} \times \underset{1}{\cancel{2}} \times \underset{1}{\cancel{2}} \times 1} \text{ or } \frac{11 \times 1 \times 7 \times 3}{1 \times 1 \times 1 \times 1} = 231$$

Volume $=$ 231 *cubic inches*

Finding the Volume of a Pyramid and a Cone

The formula for the volume of a pyramid can be derived from the formula for the volume of a prism. Similarly, the formula for the volume of a cone can be derived from the formula for the volume of a cylinder. The teacher should have the students discover the relationship between corresponding figures by the use of exploratory materials. The students should fill a container having the shape of a pyramid with sand and then empty the sand into a prism having the same base and altitude as the pyramid. He should discover that the volume of the pyramid is

Whether the cones and cylinders are made of plastics or paper, the experience of transferring volumes is a valuable one especially when the solids are related.

Stanislaus County Schools, California

one-third the volume of the prism. Similarly, he also should discover that the volume of a cone is one-third the volume of a cylinder if the bases and altitudes are equal. Thus, the formula for finding the volume of either a pyramid or a cone is $V = \frac{1}{3}Bh$. Since the base of a cone is a circle, πr^2 may be substituted for B and then the formula becomes:

$$V = \frac{1}{3}\,\pi r^2 h \quad \text{or} \quad V = \frac{\pi r^2 h}{3}$$

The formula may be written in either form because multiplying by $\frac{1}{3}$ is the same as dividing by 3.

The teacher may provide enrichment for some of the students by having them make models of pyramids and cones. Directions for making a model of a cone are found on pages 536–537.

Finding the Area and Volume of a Sphere

A basketball or a baseball is a familiar object that can be used to illustrate a sphere. A globe used in the study of geography is a well known sphere. Many students at the junior high school level are interested in the differences in the sizes of balls, such as used in baseball, golf, or tennis. These students may be interested in discovering a means for finding the surface or area of a sphere.

It is very difficult to demonstrate with objects the derivation of the formula for the surface of a sphere. The teacher is justified in having the students accept the rule for finding the area of a sphere. Meaningful learning implies that there should be "fewer rules and more reason," but not "no rules and all reason." One of the accepted procedures advocated, but seldom if ever used, in many books on the teaching of mathematics is to wind a cord about the flat surface of a *hemisphere* and about the curved surface and then to compare the length of cord used in each part. (A hemisphere is half a sphere.) In this way the relative sizes of the curved surface and the flat surface of a hemisphere can be demonstrated. To do this use a hemisphere preferably of cork. Drive a nail into the curved surface at the point at which the nail

would be perpendicular to the flat surface of the hemisphere and pass through the center of this surface. Then wind cord about the nail until the curved surface is covered and measure the cord. Next, drive the nail into the center of the circle and wind the cord about the nail until the circular surface of the hemisphere is covered. The experiment should show that approximately twice as much cord is needed to cover the curved surface as the circular surface. Since the area of a circle is πr^2, the area of a hemisphere would be $2\pi r^2$ and the area of a sphere would be $4\pi r^2$. Therefore, the formula for finding the surface of a sphere is:

$$A = 4\pi r^2$$

The teacher should not attempt to derive the formula for the volume of a sphere in the junior high school. Students in these grades have an extremely limited number of social applications for finding the volume of a sphere. The formula for finding the volume of a sphere should be accepted on authority. This formula provides excellent practice in evaluation of formulas. The formula for finding the volume of a sphere is

$$V = \frac{4}{3}\pi r^3$$

e. The Meaning of Congruence and Symmetry

Congruent Figures

Two figures are *congruent* when they match each other in full or when they *coincide*. Congruent figures have the same shape and size. In mass production all of the products that a particular machine turns out are congruent. In a shoe factory one machine may stamp out soles of the same size for a particular type of shoe. In an automobile factory one machine may stamp out left fenders for a particular model of car. The products from each of these machines are congruent. Therefore, the left fender of a particular model of a car would fit on any other car of that same model. The interchange of parts makes mass production possible. The teacher should be sure that the students understand the significance of this fact and how it affects our economic life.

Figures which have equal areas or volumes are *equivalent*. It is readily seen that two congruent figures must also be equivalent, but two equivalent figures need not be congruent. A rectangle may have an area equal to the area of a triangle, trapezoid, or another plane figure. The students should make drawings to verify the truth or falsity of the following statements or similar statements:

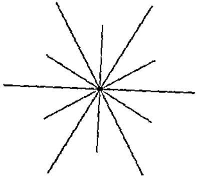
1. Two triangles having equal bases and altitudes must be both equivalent and congruent.
2. Two isosceles triangles having equal bases and altitudes must be both equivalent and congruent.
3. Rectangles having equal lengths and areas must be congruent.
4. Parallelograms having equal altitudes and areas must be congruent.
5. Regular hexagons having equal sides must be both equivalent and congruent.

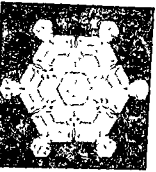
The superior student should be encouraged to discover when two figures that are equivalent must also be congruent for such familiar figures as triangles, squares, rectangles, and parallelograms.

Symmetrical Figures

Two figures that have the same size and shape but are placed in opposite positions are *symmetrical*. Symmetrical figures have *balance*. The class should discuss the meaning of this term and see if it applies to the arrangement of the items on the class bulletin board. Examples from daily life should also be given. The line or plane which divides a figure into two equal parts of the same shape is the *axis of symmetry*.

The picture of the snow crystal shows that six different axes of symmetry may be drawn in a regular hexagon. If half of the crystal is rotated about the axis forming the half, the two halves will coincide. The snow crystal represents *symmetry with respect to a line*. The students should draw the three axes of symmetry in an equilateral triangle and the one axis in an isosceles triangle.





American Museum of Natural History

Most of the class should be able to discover how many axes of symmetry are possible in rectangles, squares, and other regular polygons.

Each student should fold a sheet of paper in half. Then in the crease formed, he should place a drop of ink and press the sides of the paper before the ink dries. The design formed by the ink will be symmetric with respect to the crease in the paper.

A circle is symmetric with respect to the point known as its center. This is known as *point symmetry*. In point symmetry any diameter is an axis of symmetry. The student should be able to identify point symmetry in such familiar flowers as daisies and black-eyed Susans.

A solid may be symmetric with respect to a plane. This is known as *plane symmetry*. The significance of this kind of symmetry can be demonstrated by having a student cut an orange in half and placing wax paper between the halves. The half orange on each side of the paper is symmetric with respect to the plane. A plane which cuts a regular prism at right angles so as to form two congruent parts represents plane symmetry. The student should be especially familiar with line and point symmetry. He should be able to identify many illustrations of these types of symmetry in his environment.

The study of symmetry in mathematics at the junior high school level is predominantly for the purpose of developing an appreciation of the shape and arrangement of figures. The topic of symmetry provides almost no extension of mathematical principles or relationships. If a student is able to appreciate symmetry

and balance as it is found in nature or in structures which he sees in his environment, he will have had enriched aesthetic experiences in his daily living.

Questions, Problems, and Topics for Discussion

1. Differentiate between formal and informal geometry.
2. What is the difference between drawing a figure and constructing it?
3. Explain why no measurement can ever be exact but must be approximate.
4. How can a student discover the number of degrees there are in each angle of a regular pentagon? of a regular hexagon? of a regular octagon?
5. Make a list of the formulas for the areas of the most familiar plane figures. Write a problem illustrating each formula.
6. Suppose a student does not understand the difference between the concepts perimeter and area. Show how you would help him clarify his thinking about the meaning of these two terms.
7. Classify triangles according to sides and according to angles.
8. Make a collection of the social applications of figures which have the shape of a trapezoid.
9. Doubling the radius of a circle has what effect on the circumference? on the area? Illustrate.
10. A cubic foot of water weighs 62.4 pounds. How much would be the weight of water 1 inch deep on a surface 10 feet square?
11. With heavy waxed paper make a cylinder 7 inches in diameter and 6 inches high. Find the lateral area of the cylinder in two ways. Compute the volume of the cylinder. (Use $\pi = \frac{22}{7}$.) Show that the volume is equal to 1 gallon.
12. The side of a regular inscribed hexagon is equal to the radius of the circle. From this fact, prove that the value of π must be more than 3. (HINT: Find the perimeter of the hexagon and compare it with the circumference of the circle.)
13. The lateral area of a prism is the area of its sides. The total area of a prism is the sum of the lateral area and the area of the bases. Derive the formula for the lateral area of a rectangular prism; for the total area.
14. Write the formula for the total area of a cylinder.
15. What kind of figure is the face of a pyramid? Derive the formula for finding the lateral area of a pyramid having a square base.
16. If the outer diameter of one globe is twice the outer diameter of another globe, how do the surfaces of these two globes compare?
17. If the inner diameter of one sphere is twice the inner diameter of another sphere, how do the volumes of these two spheres compare?
18. Illustrate the difference between congruence and equivalence of figures, such as rectangles, triangles, and trapezoids.

19. In a history of mathematics look up the famous problem of "squaring a circle." Is it possible to construct a square equivalent in area to a circle?

20. Make a list of at least four items to illustrate each kind of symmetry.

21. A famous Greek philosopher stated, "God eternally geometrizes" What does this expression mean to you?

22. On the blanks below, write the correct formulas Select the correct formula from each given list·

A. Formulas for area ab, $\frac{1}{2}ab$, πr^2, $4\pi r^2$, $2\pi rh$, πrs, $6e^2$, lw, s^2, $\frac{1}{2}ps$, $\frac{1}{2}h\,(a+b)$, $2h\,(l+w)$

Formula for the area of a

1. circle:	$A =$ ____	7 rectangle	$A =$ ____
2. cone.	$A =$ ____	8. right prism	$A =$ ____
3. cube.	$A =$ ____	9 sphere	$A =$ ____
4. cylinder:	$A =$ ____	10 square·	$A =$ ____
5. parallelogram	$A =$ ____	11 trapezoid	$A =$ ____
6. pyramid:	$A =$ ____	12. triangle	$A =$ ____

B. Formulas for volume lwh, πr^2h, $\frac{1}{3}\pi r^2h$, $\frac{1}{3}Bh$, e^3, $\frac{4}{3}\pi r^3$.

Formula for the volume of a

1. cone:	$V =$ ____	4. pyramid	$V =$ ____
2. cube:	$V =$ ____	5. right prism.	$V =$ ____
3. cylinder.	$V =$ ____	6. sphere	$V =$ ____

Suggested Readings

Archer, Allene "Aids for Junior High Mathematics," *Emerging Practices in Mathematics Education*, pp 134–142. Twenty-second Yearbook of the National Council of Teachers of Mathematics. Washington, D. C. The Council, 1953.

Betz, William "The Teaching of Intuitive Geometry," *The Teaching of Mathematics in the Secondary Schools*, pp. 55–164 Eighth Yearbook of the National Council of Teachers of Mathematics New York· Bureau of Publications, Teachers College, Columbia University, 1933.

Kinney, Lucien B and Purdy, C. R. *Teaching Mathematics in the Secondary School*, pp 241–251. New York· Rinehart and Company, 1952.

Morton, Robert L. *Teaching Children Arithmetic*, pp 451–469 New York: Silver Burdett Company, 1953.

Precision—a Measure of Progress, p. 64. Detroit: General Motors Corporation, 1952.

Reeve, William D. *Mathematics for the Secondary School*, pp. 172–237. New York: Henry Holt and Company, 1954.

Chapter 12

Introduction to Algebra

A student beginning the study of algebra should understand in what ways this subject differs from and in what ways it is similar to arithmetic. The chief points of difference between the two subjects are:

1. Algebra uses general numbers, such as a or n, while arithmetic uses specific numbers. Consequently, an algebraic solution may apply to an entire class of problems while an arithmetic solution applies only to a single problem. General numbers, such as a or n are called *variables*.

2. Algebraic numbers, such as ab or b^2, do not have place value. The arrangement of letters in the term, xyz, does not affect the value of the expression, but the rearrangement of the digits in 321 would change the value of the number.

3. In algebra the number system is expanded to include numbers less than zero, such as -5. If the temperature rises 2° and then falls 2°, this fact can be expressed algebraically as $(+2) + (-2) = 0$. The zero indicates that the initial and final temperatures are the same. Hence there is no change.

The major points of similarity between algebra and arithmetic are:

1. Both are branches of mathematics dealing with quantities.

2. The fundamental operations in each are addition, subtraction, multiplication, and division.

3. Essentially the same set of principles applies to the operations of addition, subtraction, multiplication, and division in

both subjects. The reader should compare the algebraic principles listed on pages 434–436 with the arithmetic principles on pages 142–145.

The topics which have greatest social significance in algebra include: (1) the formula; (2) the equation; (3) signed numbers; and (4) the graph.

This chapter deals with the following topics:

a. The objectives of the teaching of algebra
b. Positive and negative numbers
c. Algebra as a language
d. Solution of equations
e. Solution of verbal problems.

a. The Objectives of the Teaching of Algebra

Mathematical and Social Objectives

It is generally an accepted fact that a thorough knowledge of algebra is essential to all persons specializing in the fields of engineering, the physical sciences, or mathematics. The importance of mathematics in relation to the social studies and biological sciences has increased to such an extent that algebra has become a prerequisite for students doing advanced work in these fields. However, the extent to which algebra should be mastered by persons preparing for or engaged in vocations requiring little or no mathematics is somewhat controversial. A debate on this subject would be ineffective unless an agreement could be reached concerning the following points: (1) the objectives to be pursued; (2) the content of the course; and (3) the methods employed in teaching the course.

Objectives of teaching algebra may be classified as *mathematical* and *social*. The mathematical objective is attained by preparing a student for further study in mathematics. The social objectives of the teaching of algebra must be considered very carefully when determining the status of the subject for those persons not requiring algebra as a means of earning a livelihood. Many objectives may have both mathematical and social implications as illustrated by the following:

1. *To present algebra as a language.* The formula is probably the most familiar usage of algebra as a language. Whether the formula is as simple as $A = bh$, or more complex, as $T = \pi\sqrt{\frac{a}{b}}$, it presents quantitative relationships in such a concise manner that usually the formula is much more readily interpreted in its numerical form than in its corresponding verbal form. Many non-technically trained people use algebraic formulas in their everyday lives. Without an adequate understanding of the elements of algebra, there is real danger that such formulas may be misinterpreted and incorrect results may be obtained. In fact, the formula is one means of making relatively advanced mathematics available to the layman who possesses an understanding of the basic fundamentals of algebra.

2. *To present algebra as an extension of arithmetic.* A study of elementary algebra helps to give the student a better understanding of arithmetic since the basic ideas applied in operating with algebraic expressions are the same as those used in operations in arithmetic. Thus, a study of algebra should provide an opportunity for growth in the student's ability to deal with the arithmetic processes and functional relationships required for everyday living.

3. *To present algebra as a tool for formulation of generalizations.* At the rate of 30 m.p.h., a car will travel 60 miles in 2 hours and 90 miles in 3 hours. By dealing with this situation from the algebraic point of view, it is possible to arrive at the general formula that $d = rt$. The use of letters for numbers (general numbers or variables) enables a person to write the solution for all problems dealing with rate, time, and distance. It is difficult to overestimate the value of this feature of algebra from a social or mathematical viewpoint.

4. *To present algebra as a collection of puzzles.* This aspect of algebra has been overemphasized in the past. However, because of the interest shown by many people in crossword puzzles, anagrams, and similar puzzle situations, this facet of algebra should not be overlooked. Puzzles may add to the variety of experiences available to the student. Interest instilled by this

phase of algebra helps the student to develop a healthy intellectual curiosity about the subject.

A Two-Track Program Necessary in Mathematics

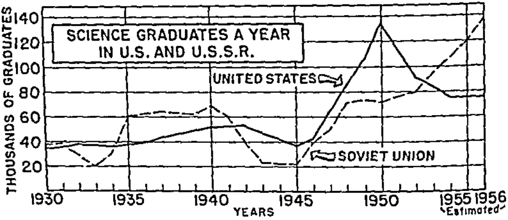
The mathematical and social phases of the objectives for teaching algebra are of approximately equal importance. A student learning algebra presented with these objectives in mind is far less likely to consider it a series of meaningless mechanical operations than the student of fifty years ago. If a year's study of algebra enables students to achieve a grasp of algebra as presented in the above objectives, this subject must then be considered as a contributing factor to general education. On the other hand, a large number of ninth grade students are not capable of completing a course in algebra because of lack of ability, background, or interest in the subject. A two-track program should be provided for such students. Frequently, students who complete a year of general mathematics in the ninth grade become interested in algebra and pursue the subject successfully. It is common practice to consider general mathematics as a course intended only for those students who cannot succeed in algebra. In many instances, general mathematics is a terminal course for those who do no further work in mathematics. Many students who are able and willing to take algebra are "guided" into general mathematics by inept counselors.

The following quotation is from an editorial in *The New York Times:*[1]

". . . many freshmen of even the highest quality frequently have had so little preparatory mathematics that science as a career is forever closed to them before they can make up their minds.

"This is a dangerous situation from the point of view of our future scientific and technological progress, and a grievous situation from the point of view of our cultural future. Great damage has been done by the old myth that mathematics is 'unimportant,' 'impractical,' and 'too hard.' School officials who have

[1] *The New York Times*, September 5, 1954



this country's welfare at heart will do well to ponder and act on the danger of the trend of making us a nation of mathematical ignoramuses at a time when our use and need for mathematics is greater than ever before in history."

This editorial presents a strong case for advising able students to continue the study of mathematics rather than discouraging them from taking this subject. These students should be advised to take at least one course in algebra as a prerequisite for further study in mathematics.

In *The New York Times*,[2] Benjamin Fine presented data to show that the U.S.S.R. is overtaking us in the vital area of training scientists and technicians. In the article he stated: "The free world is in danger of losing the important technological race for trained scientists, engineers and technicians. The Soviet Union is making an intensive effort to increase its supply of technically trained personnel.

"While the democracies of the world, including the United States, are looking the other way, the Soviet Union and its satellites are training scientists and engineers at an almost feverish pace."

The graph above is practically self explanatory. This graph shows that the number of engineers being trained in our country is decreasing, while the number of engineers being trained in the Soviet Union is increasing.

[2] *The New York Times*, November 7, 1954.

b. Positive and Negative Numbers

How to Use Signed Numbers

The ability to perform the fundamental operations with positive and negative numbers, called *signed numbers*, constitutes one of the first hurdles for the beginner in the study of algebra. A few textbooks in algebra begin the subject by introducing the meaning of signed numbers. Other textbooks introduce the subject with topics dealing with formulas and simple equations before introducing the work with signed numbers. The expansion of the number system to include positive and negative numbers marks the second major extension of the number system encountered by the student. The first major extension of the number system was that which expanded the number system to include common and decimal fractions. (Irrational numbers and square root are treated only superficially in the upper grades.) All modern textbooks in beginning algebra give excellent everyday applications of positive and negative numbers which are illustrated by the use of the thermometer, of profit and loss, and by other similar means. Not many textbooks in algebra point out that the extension including positive and negative numbers is similar in many ways to the extension including fractions.

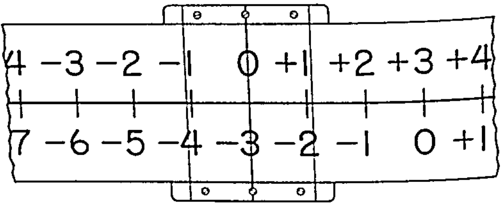
The student beginning a study of fractions presumably knows how to perform the fundamental operations with whole numbers. The student then learns how to add, subtract, multiply, and divide fractions in terms of operations with whole numbers. For instance, the process of adding fractions with unlike denominators involves multiplication, division, and addition of whole numbers. Similarly, a student beginning algebra should be proficient in the fundamental operations with arithmetic or unsigned numbers. The addition of signed numbers involves the addition or subtraction of the corresponding arithmetic or unsigned numbers. Each addition to the number system provides an opportunity for the student's growth in ability to deal with quantitative situations.

The arithmetic or unsigned number corresponding to -3 is the number 3. The arithmetic number corresponding to a signed

number is more frequently called the *absolute value* of that signed number. The numbers +5 and −5 have the same arithmetic number or absolute value, 5. Computation with signed numbers may then be reduced to performing the proper arithmetic operations with the absolute values of the numbers involved and then prefixing the proper sign. For example, the sum of +3 and −7 is obtained by subtracting the arithmetic number 3 (absolute value of +3) from the arithmetic number 7 (absolute value of −7) and prefixing to this difference a negative sign to obtain −4. The absolute values are subtracted because the numbers to be added have unlike signs. The negative sign is prefixed to the answer because the number with the larger absolute value is negative. This illustrates how to interpret the following rule given in many textbooks in algebra: *To add numbers with unlike signs, subtract the number with the smaller absolute value from the number with the larger absolute value and to this difference prefix the sign of the number with the larger absolute value.*

A Slide Rule for Signed Numbers

A slide rule, as shown below, may be constructed to perform additions and subtractions of positive and negative numbers.



To add +2 to +3, set the zero point on the slide at +3 and then proceed to +2 on the slide and read the answer of +5 on the scale below the +2 on the slide.

To add $+2$ to -3, set the zero point on the slide at -3 and then proceed to $+2$ on the slide and read the answer of -1 on the scale below the $+2$ on the slide.

To subtract $+3$ from $+2$, place the zero point on the slide at $+2$ and proceed three units to the left of this zero (the opposite of adding $+3$) to -3 on the slide and read -1 on the scale below this -3 on the slide.

To subtract -3 from $+2$, set the zero point of the slide at $+2$ and proceed 3 units to the right of this zero (the opposite direction of adding -3) to $+3$ on the slide. Read $+5$ on the scale below this $+3$.

This type of slide rule, or a similar device, is useful in demonstrating to students that principle No. 4 on page 434 of addition and subtraction is valid by working out a number of examples similar to those discussed above. Thus, a student never actually subtracts a negative number but translates a problem requiring such a subtraction into an addition problem to obtain the required answer. This is similar to the process of changing each example requiring division by a fraction to a multiplication example to obtain the required answer.

The fact that $5 \times (-4)$ is -20 may be justified by interpreting multiplication as repeated addition. If principle No. 2 on page 435 holds, then $(-2) \times (+3)$ may be expressed as $(+3) \times (-2)$ and justified in the manner stated above—as repeated additions. While it is possible to prove, with certain basic assumptions, that $(-2) \times (-3) = +6$, such proofs, according to Sanford,[3] are not very convincing to ninth grade students. It is plausible to predict that the product of $(-3) \times (-2)$ should be different from the product of $(+3) \times (-2)$. This cannot be considered as a proof that $(-3) \times (-2) = 6$, and should not be presented as such.

The inverse relationship between multiplication and division makes it necessary that the same rules for signs hold for multiplication and division. If $(-2) \times (+3) = -6$, it follows that $(-6) \div (+3) = -2$ and $(-6) \div (-2) = (+3)$ by multiplication-division principle No. 1 on page 435.

[3] Sanford, Vera "Notes on the History of Mathematics," *The Mathematics Teacher*, 244.256–257.

In the ideal situation, each student should arrive at the general rules regarding signs by first experimenting with the slide rule, and by interpreting the meaning of signed numbers in social situations involving profit and loss, left and right, and others of a similar nature.

General or Literal Numbers and Variables

In addition to becoming familiar with positive and negative numbers, the beginning student in algebra must learn how to use general or *literal numbers*. The student must learn to think of such expressions as x, $a - b$, $x^2 - y^2$, and $2mn$ as numbers. This use of letters for numbers is one of the most obvious characteristics of algebra. In elementary algebra a letter is used to represent a number in two distinct ways:

1. The letter t may be designated as a particular number required in connection with a problem. For example, in a problem involving weights, t may be designated as the number of tons of iron required. (Note that n, or x, or t must represent a number. The statement that t represents the iron required is incorrect.) Such problems are usually solved by formulating an equation from the conditions given. This equation is then solved for the unknown number. Such an equation is called a *conditional equation*.

2. The letter y, as used in $y + 2y = 3y$, represents any number. In this case the equality is not solved for y. This algebraic statement shows a relationship that is true for all values of y. An equality of this type frequently is called an identical equation or an *identity*.

It is helpful for the beginner to recognize the difference between these two uses of letters for numbers. This difference is discussed more fully on page 429 in the section on *Algebra as a Language*.

Major Problem in Teaching Algebra

One of the most difficult problems in the teaching of algebra is to obtain a reasonable balance between interpretation and

manipulation. Probably one of the most common errors in the past has been an overemphasis on manipulation. On the other hand, without a sound knowledge of fundamental operations, the student will have little to interpret. The remainder of this chapter is devoted to explaining various ways of interpreting algebraic processes.

The Function Concept

If two quantities are related so that the value of one quantity depends upon the value of the other, these quantities are said to be *functionally related*. Distance is a function of rate and time. The area of a circle is a function of the radius. The perimeter of a square is a function of the length of one of its sides. The cost of a number of tons of coal is a function of the number of tons and the cost per ton.

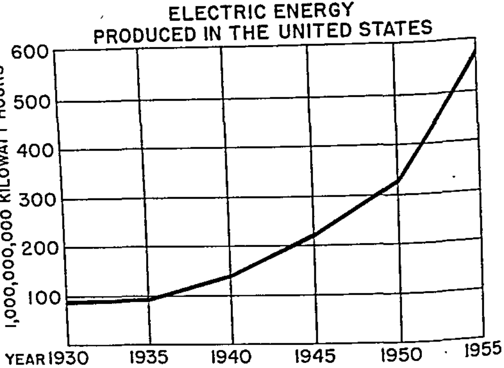
Functional relationships in elementary algebra are described in four different ways:

1. *A function may be described verbally.* Distance is expressed verbally as a function of rate and time in the paragraph above. This type of representation is useful for purposes of discussion and for describing simple relationships but is probably the least useful of the four ways of expressing such a relationship.

2. *A function may be described by a formula.* Most of the functions encountered in elementary algebra may be described by a formula, but many functional relations are too complex to be represented by a formula.

3. *A function may be described by a table.* Bankers usually use tables to determine interest on loans. Considerable time is required to construct such tables but once constructed they are time-saving devices. Tables of functional relationships are frequently used in the fields of finance, engineering, and mathematics.

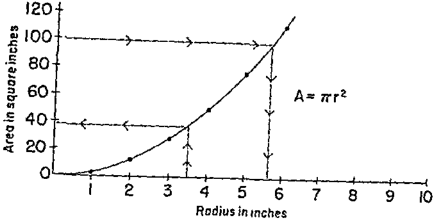
4. *Functional relationships may be represented graphically.* Graphs often are used by editors of magazines and newspapers to present functional relationships to the general public because such graphs make certain phases of the function more apparent than



any other type of representation. Consider the following examples:

a. The graph above shows the functional relationship between time, for the interval from 1930 to 1955, and electric energy produced in the U. S. A glance at the graph indicates that the amount produced each year has risen sharply over the entire interval. No other form of representation could demonstrate this fact so vividly. While a line graph is used above, other types, such as a bar or pictograph, could be used.

b. The graph on page 420 shows the functional relationship between time and the number of scientists produced by the United States and by the U.S.S.R. The graph shows that the United States produced more scientists than the U.S.S.R. in 1950, and that the U.S.S.R. produced more scientists than the United States in 1954. These facts are obtained and recognized more easily from the graphical form of representation than from any of the other forms.



c. The graph above shows the relationship between the area of a circle and the length of its radius. By following the dotted lines in the direction of the arrows, one may obtain the following information:

The area of a circle having a radius of 3.5 inches is approximately 38 square inches.

The radius of a circle having an area of 100 square inches is approximately 5.6 inches.

Each of these methods of describing a function has its advantages and disadvantages. The verbal form might bring out phases that otherwise might be overlooked but this form may be too long and involved in many instances. The formula is a representation of a quantitative situation but frequently it is difficult to obtain numerical data by use of a formula. The tabular representation of a function is excellent when the frequency of use justifies the initial labor necessary to construct the table. Such tables frequently do not include all values required for practical work. Graphs provide an effective means for presenting the highlights of a function such as the highest and lowest values and areas of rapid change. The data obtained from most graphs, however, usually are not as accurate as the data obtained from the second and third types of functional representations. The beginner in algebra not only should become familiar with the various ways of describing functions but also should learn the advantages and disadvantages of each.

c. Algebra as a Language

Translating Statements into Algebra and Vice Versa

Algebra is treated as a language in most current texts, but there are ways in which the treatment of this important topic may be improved. Problems of the following types are found in practically every current text in elementary algebra:

Write algebraically:

1. a less than x
2. b more than y
3. t less than the sum of m and n
4. George's age, if George is n years older than Ruth and Ruth's age is y years.

To be successful in algebra, the student must acquire skill in translating verbal statements into the language of algebra. Many students cannot solve word problems because of lack of ability in this skill. Practice should be given at regular intervals in order that students may maintain and improve their ability in this field.

The language of algebra will probably be better understood if algebraic statements are translated into verbal statements almost as frequently as verbal statements are translated into algebraic statements.

The following examples illustrate how such translations indicate to a great degree the extent of a student's ability to interpret algebra.

1. Algebraic statement: $\frac{1}{2}x + \frac{1}{2}x = x$

   First translation: One-half of x plus one-half of x is equal to x. The student making this translation indicates ability to transmit the statement verbally but does not indicate any understanding.

   Second translation: One-half of any number plus one-half of the same number is equal to the number. The student making this translation recognizes that the statement is an identity but the statement is not concise.

   Third translation: Two halves make a whole. This non-literal translation is probably the best because of its conciseness.

2. Algebraic statement: $x - 3 = 5$

First translation: x minus three equals five. The student making this translation gives a verbal translation of the equation but does not indicate any understanding of the meaning of the statement.

Second translation: Some number minus three is equal to five. The student giving this translation recognizes that the equation is conditional.

Third translation: Three less than what number is equal to five? The student giving this translation recognizes the challenge contained in each conditional equation—the challenge to solve the equation.

Identities

The symbol $\equiv$ is the identity symbol. Two expressions are identical if they are equal for all permissible values of the literal number or numbers involved. $2 + 3 \equiv 5$ is an identity. $2x + 3x \equiv 5x$ is an identical equation. It is common practice to refer to both as identities. It is also correct to write these equations with an equal sign rather than an identity sign.

The equation $x + 2 = 4$ is a conditional equation. It is usually referred to as an equation. In elementary algebra the word equation, unqualified, almost invariably refers to a conditional equation which presents the challenge of determining for what number or numbers the equation is true.

Students should soon learn to distinguish between identities and conditional equations. The addition, subtraction, multiplication, and division of algebraic expressions produce identities. The process of factoring is another means of forming identities. Translation from algebraic to verbal language should help the student to recognize the difference between identities and conditional equations as illustrated in the examples above. In some instances in the past, the students have not been introduced to the word, identity, until they begin the study of trigonometry. There is no justification for such a delay in the introduction of this term or of this concept.

The Formula as a Means of Showing Quantitative Relationships

The formula as a record of important quantitative relationships (functional relationships) is probably the most common use of the language of algebra in everyday affairs. Appreciation of the value of the language of algebra may be increased by comparing the convenience of working with formulas in the algebraic and verbal forms.

The student's first experience with algebra usually involves the work dealing with formulas which is encountered in the seventh and eighth grades. Proper introduction of this work pertaining to formulas will help to provide a sound foundation for further work in algebra. This work should help the student to make the following generalizations:

1. A formula is an expression, in algebraic language, of the relationship between two or more quantities which can be expressed numerically.
2. Algebraic language may be used more conveniently and precisely for representing such relationships than verbal statements.
3. The letter w (or l), as in the formula $A = lw$, represents any number that may be used to describe the width (or length) of a rectangle.
4. Formulas are worthless unless the algebraic language is interpreted correctly. The student must understand that Bh indicates that the number B must be multiplied by the number h, or that $\frac{a}{b}$ indicates that the number a must be divided by the number b.
5. The ability to perform the indicated arithmetic operation of a formula is of little practical value unless the student has some knowledge of the types of units required. If a rectangle has dimensions of 3 inches by 4 feet, a student who multiplies 3 by 4 to get the area understands neither the use nor the meaning of the formula.

The following verbal translations of the formula $A = lw$ show the difference between a limited interpretation and one which is fully understood.

a. The area of a rectangle equals its length times its width.
b. The *number of square units in the* area of a rectangle is equal to the product of its length and width, both measured in the linear units corresponding to the square unit used to designate the area.

Reference to the formula $E = mc^2$ appears frequently in newspapers. Relatively few people can give the following translation: If a given quantity of matter is transformed into energy, the number of dynes of energy obtained is equal to the product of the number of grams of matter involved and the square of the velocity of light measured in centimeters per second.

The Subject of a Formula

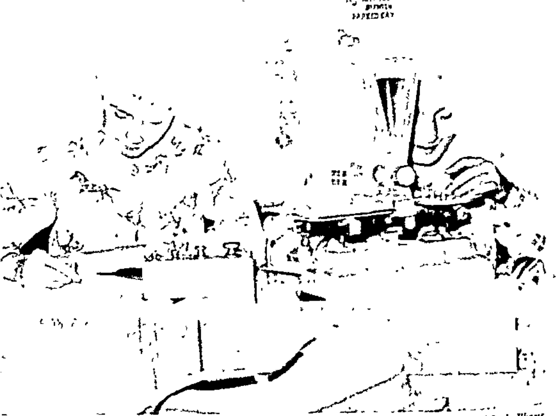
In the formula $A = bh$, A is called the *subject of the formula.* Elementary work with formulas usually consists in determining the subject of the formula when numerical values for all the other quantities involved are given. After the student has had an introduction to equations, he should realize that a formula has many more uses than the function of determining the value of its subject. A formula may be used to determine the value of any of the quantities involved when the numerical values of each of the remaining quantities are known. In the solution of the following problem, the value of b in the formula $A = \frac{1}{2}h\,(a + b)$ may be determined if given numerical values for A, h, and a.

The area of a trapezoid is 90 square inches. What is the length of its upper base if its altitude is 6 inches and its lower base is 19 inches?

Solution: $A = 90$ sq. in.; $h = 6$ in.; $a = 19$ in.

Begin with:	$A = \frac{1}{2}h\,(a + b)$
Substitute values:	$90 = \frac{1}{2}\,(6)\,(19 + b)$
Combine:	$90 = 3\,(19 + b)$
Divide both sides by 3:	$30 = 19 + b$
Subtract 19 from both sides:	$11 = b$

The length of the upper base must be 11 inches.



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· Experimentation with equivalence of weights can lead to the formulation of a formula.

Formulas as Social Applications of Mathematics

While formulas play an important part in introducing the student to algebra as pure mathematics, formulas are also very useful in the social applications of mathematics. There are many opportunities to use formulas in everyday life. It is quite common to find newspaper or magazine articles discussing the use of formulas for a great variety of topics.

The AAA recently published a formula[4] for finding the approximate cost of driving a car. The verbal statement of this formula was: The cost of driving a car is $1.55 for every day the car is driven plus 3.5¢ a mile for each mile the car is driven. If the car is driven over 18,000 miles per year, $10.56 should be added to the cost for each 1,000 miles over 18,000 miles. The reader should translate this verbal statement into the language of algebra.

[4] See automobile unit in Chapter 8, pages 291–295.

A fireman uses the formula, $D = \frac{h}{5} + 2$, to set a ladder for the greatest safety. In this formula, D represents the horizontal distance from the base of the wall on which a ladder rests to the foot of the ladder and h represents the length of the ladder.

Students should be encouraged to develop a variety of formulas that can be used in their own everyday affairs. A few such possibilities are:

1. Develop a formula which will designate the starting time of each period of the school day. T (time) is a function of n (the number of periods).

2. Develop a formula to represent how much money would be collected by the sale of tickets to adults and students for a school play. S (total money) is a function of s (the number of student tickets) and a (the number of adult tickets).

Further suggestions for interpreting the formula are given on page 438 in the section on *Algebra as a Tool for Generalization*.

Students should be encouraged to examine textbooks in science and mathematics and handbooks for given professions to find formulas which seem interesting to them. It is an excellent project for a student who discovers an interesting formula to present its derivation and some of its applications to the class.

Students also should investigate to determine if their parents make use of formulas in their profession. Skilled workmen in many trades frequently have useful rules for various phases of their work which may be expressed verbally or algebraically.

A thorough treatment of algebra as a language with repeated translations to and from verbal forms can do much to help students to recognize that algebra is much more than a series of mechanical manipulations. Proper emphasis on the social aspect of formulas can do much to present algebra as a practical subject.

Algebra as an Extension of Arithmetic

A set of principles for arithmetic is given on pages 142–145. The more able students usually discover many of these independently. If a student understands these principles governing

algebraic operations, he should discover that algebra is an orderly subject, comparatively easy to learn. Manipulative algebra is, essentially, the application of these principles to algebraic expressions.

From the result of several years of experimentation, the writers formulated the following set of principles. These principles have proven a valuable aid for providing explanations why the fundamental operations involving addition, subtraction, multiplication, and division are performed. Each day in the classroom opportunities are offered to refer to one or more of these principles. The reader may compare the advantages and disadvantages of subdividing these principles into two groups instead of four. Since positive and negative numbers are not normally used in arithmetic, principle No. 4 may not be traced directly to the arithmetic principles on page 142. The study of algebra provides a valuable means of broadening the student's understanding of these principles.

Addition and Subtraction Principles

1. *Addition and subtraction are inverse operations because addition may be nullified by subtraction and vice versa.* If the number x is increased by 2 to obtain $x + 2$, the original number x may be obtained by subtracting 2 from $x + 2$. If 3 is subtracted from the number t to obtain the difference $t - 3$, the original number t may be obtained by adding 3 to $t - 3$.

2. *Only like quantities may be combined by addition or subtraction.* Thus, $2x$ and $3x$ may be combined by addition to obtain $5x$, but the sum of $2x$ and $3y$ must be indicated as $2x + 3y$.

3. *A change in the order of adding numbers does not change the sum, but a change in the order of subtracting one number from another does change the difference.* The student must understand the principle governing the subtraction of two numbers. The student who writes $b - a$ in place of $a - b$ violates this principle. Most students apply the addition phase of this principle quite readily.

4. *Instead of subtracting a number, one may add the negative of that number, or instead of adding a number, one may subtract the negative of that number.* Instead of subtracting A, it is possible to add $-A$;

instead of adding B, it is possible to subtract $-B$. This is a generalization of the rule of subtraction which states, "Change the sign of the subtrahend and add."

The concept of the negative of a number is useful. The negative of a number may be found by multiplying the number by -1. Each of the following pairs gives a number and its negative: $3, -3$; $-5, 5$; $2x, -2x$; $-5ab, 5ab$; $a - 2b, -a + 2b$ or $2b - a$.

Principle No. 4 provides a very satisfactory answer to a question that proves troublesome to students in beginning algebra. Is $a - b$ the difference of $(+a)$ and $(+b)$ or the sum of $(+a)$ and $(-b)$? Either interpretation is correct, since the subtraction of $(+b)$ is identical with the addition of $(-b)$. There is an advantage in interpreting $a - b$ as a sum. When interpreted as a sum, the order of addition may be changed to obtain $-b + a$. No such flexibility exists when $a - b$ is interpreted as a difference. Consequently, the verbal statement of the expression, $5x - 2y - 3$, should be $5x$ plus $(-2y)$ plus (-3).

The use of parentheses also may be explained by principle No. 4 since $a - (b - c)$ may be written as $a - b + c$ because $-b + c$ may be added instead of subtracting $b - c$ Similarly, $a + x - y$ may be written as $a - (-x + y)$ since $-x + y$ may be subtracted instead of adding $x - y$.

Multiplication-Division Principles

1. *Multiplication and division are inverse operations since either operation may be used to nullify the other.* When x is multiplied by 2 to obtain $2x$, the original number x may be obtained by dividing $2x$ by 2. When y is divided by 3 to obtain $\frac{y}{3}$, the original number y may be obtained by multiplying $\frac{y}{3}$ by 3.

2. *Changing the order of multiplication does not change the value of the product, but changing the order of division does change the value of the quotient.* The change in order of multiplication is usually done readily. It is not uncommon in the example, $y \div x$, to get an answer of $\frac{x}{y}$ in place of $\frac{y}{x}$ from a student beginning the study of algebra.

3. *To multiply (or divide) a polynomial by a number, multiply (or divide) each term by that number.* A polynomial is an algebraic expression of two or more terms. This principle applies in arithmetic when multiplying 34 by 2 and in algebra when multiplying the polynomial $3a + b - c$ by 2.

4. *To multiply (or divide) a product by a number, multiply (or divide) one factor by that number.* To multiply $3(a + b)$ by 2, one may multiply either the 3 by 2 or the $a + b$ by 2. Much of the manipulative difficulty in elementary algebra is due to failure to distinguish between *terms* and *factors* and the lack of understanding of multiplication-division principles No. 3 and No. 4. Terms in algebra are synonymous with addends in arithmetic. Factors are numbers which indicate products.

5. *To divide by a number, multiply by the reciprocal of that number; to multiply by a number, divide by the reciprocal of that number.* This principle explains why $\frac{1}{2}a$ and $\frac{a}{2}$ are the same because multiplying by $\frac{1}{2}$ and dividing by 2 produce the same results. This principle is a generalization of the rule, "Invert the divisor and multiply."

6. *The value of a fraction is not changed when the numerator and denominator are multiplied (or divided) by any number except zero.* This principle is fundamental in the treatment of fractions and should be referred to frequently. There should be ample opportunity to demonstrate this principle in dealing with fractions. This principle will rarely be used without involving one or more of the other multiplication-division principles, as shown in the following example:

$$\frac{2a + 2b}{6ab} = \frac{a + b}{3ab}$$

The fraction is reduced by dividing both numerator and denominator by 2. To divide the numerator by 2, each term must be divided by 2. To divide the denominator by 2, only one factor, 6, must be divided by 2. Thus, multiplication-division principles No. 3, No. 4, and No. 6 are involved in reducing one fraction to its lowest terms.

Use of Principles Governing Fundamental Processes

By frequent references to these principles, the seemingly unrelated mechanical manipulations of algebra are given a basic structure. The following suggestions are given for dealing with these principles:

1. This set of principles is not the only effective set possible. The set used by any teacher should be constantly scrutinized for improvement by additions, deletions, rewording, or reorganization. Comparison with the set on pages 142–145 will give some notion of how variations may be made.

2. Do not insist that students memorize the principles. In the ideal situation, the student should discover each principle for himself. In practice, every question pertaining to basic operations should be answered in terms of the principles. The writers have found that this set is adequate for answering all questions pertaining to addition, subtraction, multiplication, and division which normally occur in elementary algebra. The work with equations, exponents, and radicals needs additional principles.

3. It is most important that the student of algebra become familiar with a set of principles similar to those on pages 142–145 which he has learned in arithmetic. The student should then recognize that practically the same principles he has learned in arithmetic are used in algebra. Whenever possible, elementary and junior high teachers should consult with each other to insure that the principles used by each are compatible.

Algebra as a Collection of Puzzles

While this phase of algebra can be and has been overdone, it should not be neglected. Many people will spend hours attempting to work a puzzle when they would resent one-half hour spent on a systematic treatment of the same subject. A teacher should have a collection of puzzles to be introduced at appropriate times. While it is desirable that the puzzle should have some bearing on the work at hand, it would not be necessary. A puzzle may provide a needed break in the routine or provide a good introduction to future work.

Algebra as a Tool for Generalization

The phase of algebra which produces general answers may be one of its most valuable uses. Since algebra is considered by many to be theoretical and useless in everyday affairs, any practical feature of it should be emphasized. There is ample opportunity to demonstrate this feature of algebra in the traditional work of the ninth grade.

The following sequence of problems illustrates how a simple generalization may be reached.

Question: How far does a car travel in 7 hours at 60 m.p.h.?

Answer: (7×60) miles, or 420 miles.

Question: How far does an object travel in 35 seconds at 20 ft./sec.?

Answer: (35×20) ft., or 700 ft.

Question: How far does a plane fly in t hours at r m.p.h.?

Answer: $(t \times r)$ miles $=$ tr miles, or rt miles.

In the last instance, by using letters for numbers, the problem has been solved for all cases and the formula, $d = rt$, has been obtained.

The following problems require varied techniques:

What is the cost of n apples if each apple costs r cents?

If b oranges cost k cents, what is the cost of one orange?

If y oranges cost c cents, what is the cost of z oranges?

If t tons of one grade of coal cost r dollars and n tons of a lower grade cost s dollars, what is the cost per ton of each grade? Express as a single fraction the difference in cost between a ton of the first grade and a ton of the second grade.

The answer to each of the above problems is expressed as a formula which will solve all problems of that particular type.

Problems of this type should be preceded by corresponding problems using numerical numbers rather than literal numbers. Such procedures help to emphasize some practical applications of the use of a letter, such as n, as a general number. When this concept has been established, the general solutions obtained may then be applied to numerical situations. A teacher should always

be alert to find new situations which are suitable for developing formulas. The closer these situations are to the student's experience, the more useful they are in making the student recognize the value of algebra.

An alert teacher can stimulate students' interests by dealing with problems which function in their lives. The following sequence of problems shows how to make problems dealing with the cost of travel by car.

1. On the open highway, car A runs 15 miles on a gallon of gasoline and car B runs 18 miles on a gallon. If gasoline costs 30¢ per gallon, how much more will the gasoline cost per mile for car A than for car B?

Solution: Assume a trip of 270 (15×18) miles is made since this number is divisible by both 15 and 18.
In 270 miles, car A will use 18 gallons of gasoline.
In 270 miles, car B will use 15 gallons of gasoline.
In 270 miles, car A uses 3 gallons more than car B.
In 270 miles, gasoline for car A costs 90¢ more than that for car B. (3×30¢)
In one mile, gasoline for car A costs approximately .333¢ more than that for car B. (90¢ $\div$ 270)

2. Car A runs t miles on a gallon of gasoline when traveling and car B runs n miles on a gallon. If gasoline is sold for c cents per gallon, how much more will the gasoline cost per mile for car A than for car B? This implies that t is less than n.

Solution: Assume a trip of nt miles is made since this number is divisible by both n and t.
In nt miles, car A uses n gallons of gasoline.
In nt miles, car B uses t gallons of gasoline.
In nt miles, car A uses $n - t$ gallons of gasoline more than car B.
In nt miles, gasoline for car A costs $(n - t)\ c$ or $c\ (n - t)$ cents more than that for car B.
In one mile, gasoline for car A costs $\dfrac{c\ (n - t)}{nt}$ cents more than that for car B.

3. George's car uses gasoline at the rate of 16 miles per gallon and Jane's car uses gasoline at the rate of 19 miles per gallon. How much greater is the cost of gasoline per mile for George's car than for Jane's car if gasoline costs 26¢ per gallon?

Solution: $n = 19$; $t = 16$; $c = 26$; $D = ?$

$$D = \frac{26\ (19 - 16)}{19 \times 16} = .257$$

Therefore, the cost of gasoline for George's car is .257¢ per mile more than that for Jane's car.

4. John's car is now using gasoline at the rate of 20 miles per gallon. How many more miles per gallon are necessary to reduce the cost $\frac{1}{4}$¢ per mile if gasoline sells for 30¢ per gallon?

Solution: Let r be the required increase in miles per gallon.
$n = 20 + r$; $t = 20$; $D = \frac{1}{4}$; $c = 30$; $r = ?$

$$\frac{1}{4} = \frac{30\ (20 + r - 20)}{20\ (20 + r)}$$

Solving for r: $r = 4$

An increase of 4 miles per gallon from 20 miles per gallon to 24 miles per gallon will reduce the gasoline cost $\frac{1}{4}$¢ per mile.

In this sequence, problem 1 may easily be omitted if the class is sufficiently able to think with general numbers. Otherwise, it may be necessary to give a sequence of such problems so that the arithmetic of the situation is well in mind before the algebraic or general solution is attempted.

Problem 2 demonstrates the use of algebra in solving all problems of a class by using general numbers (letters) instead of arithmetic numbers. The formula developed, $D = \frac{c\ (n - t)}{nt}$, represents the solution of all such problems.

Problem 3 is no different in type from problem 1. However, problem 3 may now be solved readily by the formula developed in problem 2.

Problem 4 demonstrates that the formula developed is not limited to determining the value of the subject D when n, t, and c are known. This formula makes it possible to determine any one of the four quantities involved when the other three are known. This phase will usually be understood only by the more able students.

This sequence may be summarized in the following four steps:

1. Arithmetic solutions
2. The general solution
3. The application of the general solution to numerical situations
4. Extensions of the application of general solutions for gifted students.

This sequence should be repeated many times with different practical situations. An examination of the traditional approach to quadratic equations will reveal this sequence. Thus, the use of algebra to obtain general solutions may be demonstrated in a first course in a way that helps students to recognize that algebra is a useful tool in solving problems in practical affairs.

d. Solution of Equations

The Axioms Governing the Solution of Equations

The solution of an equation in algebra makes use of the principle or axiom of balance. This axiom states that the performance of an operation (such as addition) on one side of an equation will preserve the original equality only if the same operation is performed on the other side. Sometimes this principle is broken down into four axioms, one each for addition, subtraction, multiplication, and division. The addition axiom may be stated: If equals are added to equals the sums are equal. The remaining axioms follow the same pattern except the division axiom which excludes division by zero. A visual representation of this idea is aided by the analogy of a balance scales. An equation may be compared with a set of scales in balance. If two pounds are removed from one side of the balanced scales, two pounds must be removed from the other side if balance is to be maintained.

Similarly, if 2 is subtracted from the left side of the equation, $3x + 2 = x + 4$, 2 must also be subtracted from the right side if the two expressions are to remain equal in the form, $3x = x + 2$.

When the property of balance is established, there still remains the problem of deciding which operation to apply to both sides. The following equations may be used to demonstrate how the first principles in addition-subtraction and multiplication-division provide the answer to this question.

1. $x + 3 = 7$
2. $x - 5 = 9$
3. $5x = 35$
4. $\frac{x}{3} = 5$

To solve equation 1, the addition of 3 on the left member may be nullified by subtracting 3. To maintain balance, 3 must also be subtracted from the right member.

To solve equation 2, the subtraction of 5 on the left may be nullified by adding 5. To maintain balance, 5 must also be added on the right.

To solve equation 3, multiplication by 5 on the left may be nullified by dividing by 5. Balance must be maintained by dividing by 5 on the right.

To solve equation 4, division by 3 on the left may be nullified by multiplying by 3. Balance must be maintained by multiplying by 3 on the right.

The four equations above are called *first degree equations* because only the first power of the unknown occurs and the unknown does not occur in the denominator of any fraction involved. Because the graphs of such equations are straight lines, they are frequently referred to as *linear equations*. All first degree equations may be solved by one or more operations of the types illustrated above. It is frequently desirable to simplify one or both sides of an equation before operating on both sides. Simplification is obtained by using algebraic identities. In most equations of elementary algebra, these identities result from simple additions, subtractions, multiplications, or divisions. The solution of the

following equation demonstrates the use of both the principle of balance and identities.

5.

$$
\begin{array}{lll}
 & & 3(x-1)+x = 2x-1 \\
\text{Identity:} & 3(x-1) \equiv 3x-3 & \\
 & & 3x - 3 + x = 2x-1 \\
\text{Identity:} & 3x + x \equiv 4x & \\
 & & 4x-3 = 2x-1 \\
\text{Subtract } 2x \text{ from both sides:} & & 2x-3 = -1 \\
\text{Add 3 to both sides:} & & 2x = 2 \\
\text{Divide both sides by 2:} & & x = 1
\end{array}
$$

$$
\text{Check: } 3(1-1)+1 = 2-1
$$
$$
1 = 1
$$

The operation to be performed in each succeeding step should be indicated in a manner similar to that shown above The student is more likely to understand an operation if he can indicate the nature of this operation.

The class should discuss the verbal translations of equations. The translation of equation 1 would be: "Three more than what number is equal to seven?" The translation for equation 5 given above might be: "Three times 1 less than what number plus the number is equal to 1 less than twice the number?"

It should be standard practice to check equations. There are two reasons for checking equations:

1. Equations should be checked to insure, as nearly as possible, that no error has been made.

2. Equations should be checked to insure that the answer obtained is a solution. In the following equation, correct procedures will indicate that the solution is $x = 1$.

$$
\frac{3}{x-1} = \frac{x+2}{x-1} - 3
$$

When 1 is substituted for x in the equation, division by zero is required. Since division by zero is impossible, this equation has no solution.

Two common methods for the solutions of equations involving fractions are illustrated on the next page.

A.

$$\frac{2x}{3} + \frac{1}{2} = \frac{11}{6}$$

Multiply both sides by 6:	$4x + 3 = 11$
Subtract 3 from both sides:	$4x = 8$
Divide both sides by 4:	$x = 2$

Check: $\frac{4}{3} + \frac{1}{2} = \frac{11}{6}$

$\frac{11}{6} = \frac{11}{6}$

B.

$$\frac{2x}{3} + \frac{1}{2} = \frac{11}{6}$$

Identity: $\frac{2x}{3} + \frac{1}{2} = \frac{4x + 3}{6}$

$$\frac{4x + 3}{6} = \frac{11}{6}$$

Multiply both sides by 6:	$4x + 3 = 11$
Subtract 3 from both sides:	$4x = 8$
Divide both sides by 4:	$x = 2$

The check is the same as above.

In A, all fractions are eliminated in the first step by multiplying both sides of the equation by the least common denominator of the fractions occurring in the equation. This procedure may be followed for all cases except those involving complex fractions. After multiplying by the L.C.D., the remainder of the solution follows by the methods previously indicated. The only operation with fractions required by this method is the multiplication of a fraction by a whole number.

In B, the first step requires the addition or subtraction of fractions. The second step requires the multiplication of fractions by a whole number. The remainder of the solution follows as in A. The method in B may be used to give additional practice in addition and subtraction of fractions but this reason alone would be somewhat artificial.

Solution of Proportions by Algebra

A proportion is a statement expressing the equality of two ratios or fractions. (See page 347.) When the proportion contains an unknown, it becomes a fractional equation and should be solved as such. To solve the following proportion, multiply both sides of the equation by the least common denominator of the fractions involved.

$$\frac{3}{x} = \frac{8}{40}$$

Multiply both sides by $40x$:	$120 = 8x$
Divide both sides by 8:	$15 = x$

All proportions should be solved in this manner in elementary algebra. The shortcut of cross multiplying saves little time and should not be introduced until the fundamental approach just indicated is thoroughly understood.

Factoring

Factoring may be defined as the process of replacing a performed multiplication by an equivalent indicated multiplication as follows:

$$x^2 - y^2 \equiv (x + y)(x - y)$$

The most common reason for factoring is to make division easier. Ability to factor is an asset in the simplification of many algebraic expressions, as shown in the following example.

$$\frac{2x^2 - 14x + 24}{4x^2 - 36} = \frac{2(x^2 - 7x + 12)}{4(x^2 - 9)} =$$

$$\frac{2(x - 3)(x - 4)}{4(x + 3)(x - 3)} = \frac{x - 4}{2(x + 3)}$$

Today there is a strong movement to reduce drastically the amount of factoring in elementary algebra. There is ample justification for such a movement when factoring is taught as a purely mechanical operation. In extreme cases over a dozen

different types of factoring have been presented in beginning courses in algebra. The number of types of factoring to be covered is not as important as the proper interpretation of the nature and importance of factoring. This is one phase of algebra in which manipulation and interpretation have been in imbalance.

Three Important Types of Factoring

An examination of the current texts in elementary algebra will find an almost unanimous agreement on the following three types of factoring:

1. The removal of a common monomial factor:
$$x^2 - 3x \equiv x(x - 3)$$
2. The difference of two squares: $a^2 - b^2 \equiv (a + b)(a - b)$
3. A trinomial expressed as the product of two binomials:
$$n^2 - 3n + 2 \equiv (n - 1)(n - 2)$$

It should be noted that the identity sign, $\equiv$, is used in place of the equal sign in each of the above examples. While this practice is not essential, it is useful in helping to remind the student of the distinction between a conditional equation and an identity.

By pointing out the following features of factoring at appropriate times, much can be done to increase the student's appreciation of his operation:

1. Factoring is an inverse process of multiplication, since it nullifies a previous multiplication. Factoring is correct when the performance of the indicated multiplication will produce the original expression.

2. Each completed factoring exercise should be recognized as an algebraic identity which has produced two expressions that appear to be different but have the same numerical value for all permissible values of the variable (or variables) involved. The letters in an identity are frequently referred to as variables since these letters may take many different numerical values.

3. The identity or equal sign is a two-way indicator. In the identity $x^2 - y^2 \equiv (x + y)(x - y)$, it should be understood that the expression on the left may be replaced by the expression

on the right and vice versa. Usually the expression on the left would be more useful in addition and subtraction while the expression on the right would be more useful in multiplication and division.

e. Solution of Verbal Problems

Essential Skills Involved in Problem Solving

The reader should review Chapter 9 on problem solving. The principles stated in that chapter apply to algebra. The ability to solve word or verbal problems in algebra depends upon two major skills:

1. The ability to recognize relationships by analyzing situations.

2. The ability to express these relationships in algebraic language.

The first skill is difficult to develop. It is acquired principally through experience in analyzing problems The second skill may be developed by means of special exercises, such as those on page 428. Several common practices, now in frequent use, are detrimental to the development of skill in analyzing situations. They tend to enable the student to work problems with a minimum of thinking instead of developing the power to think for himself.

One practice that is detrimental is to place too much emphasis on types of word problems. By spending several consecutive lessons on a particular type of problem, such as mixture problems, the student is apt to form generalizations applicable only to the one type of problem. These generalizations will be more harmful than helpful for other types of problems. To the extent that a student solves all mixture problems by rule of thumb, to that extent he fails to acquire skill in analyzing situations, *unless the mechanization has been achieved as a result of the student's own analysis.*

Another practice which is harmful in expressing numerical relationships in algebraic language is to overemphasize the importance of the answer. Too many students obtain the

impression early in their academic life that any way of getting the correct answer is justifiable. Students should learn that sound methods produce correct answers when there are no errors in computation. Students should also recognize that a method which produces correct results only *some of the time* cannot be tolerated. While it is sound practice, under some conditions, to give a student partial credit for a problem that has been correctly analyzed but which contains computational errors, the student should never be given the impression that such errors are unimportant. When mathematics is applied to actual situations in the practical world, an error in computation is just as serious as an error in method.

A third unfavorable practice is to use boxes or special tables to mechanize solutions. The following problem frequently is solved with the aid of a box: How much 50% glycerine solution must be added to 10 quarts of a 20% solution to make a 30% solution?

Solution of problem: Let x equal the number of quarts of 50% solution needed

	Quarts of solution	Quarts of glycerine
Initial quantity	10	2
Quantity after adding x quarts of 50% solution	$10 + x$	$2 + \frac{1}{2}x$

Since 30% of the final solution must be equal to the final amount of pure glycerine:

$$.3\,(10 + x) = 2 + .5x$$

Solving for x: $x = 5$

Therefore, 5 quarts of 50% solution are needed.

If the above box is used as a means of organizing information, it may be approved as an instructional aid in problem solving. If the box is a mechanical device to avoid thinking, then the representation defeats the purpose of presenting the problem.

A Major Difficulty with Verbal Problems

The solution of verbal problems causes a major difficulty as too many problems lack reality to the student. It is difficult to obtain a student's interest in problems which seem artificial. This difficulty can probably never be completely eliminated, but a teacher should be guided in the choice of problems by the interest demonstrated by the class. A teacher should accumulate problems to supplement those of the text to serve a wide variety of interests.

Problems demanding the derivation of fairly simple formulas for everyday situations, such as those on page 438, have probably been neglected too much in comparison with the emphasis on word problems requiring the formulation and solution of an equation.

The following suggestions may be helpful:

1. There should be a limited number of problems of the same type in any one exercise.

2. Every effort should be made to guard against routine and mechanical solutions. The ability to analyze is more important than the ability to follow instructions.

Word problems demand the interpretation of algebra as a language. They give the student the opportunity to discover that algebraic manipulations are not meaningless but are useful in obtaining practical results.

In solving a problem in which t is the unknown, t should always be labeled as a number. Instead of stating that t equals the money, it should be stated that t is the number of dollars required. This practice will help stress the fact that t can be only a number.

3. A teacher should always be alert and look for problems which may interest the class more than those in the text.

4. A review of the arithmetic background for a given type of problem will often be of great help.

Sequential Nature of Algebra

There is a definite sequence of ideas and operations in algebra as in arithmetic. Inability to add will make it impossible to

perform multiplication. Similar sequences are too numerous to mention. The most complicated operation may be reduced to a series of simple operations. The following device, called a ladder, highlights this fact.

Complete the following operations:

1. The quotient of x^2 divided by x is ________.
2. The product of x and x is ________.
3. The product of x and -3 is ________.
4. The difference of x^2 and x^2 is ________.
5. The difference of $-3x$ from $-4x$ is ________.
6. The quotient of $-x$ divided by x is ________.
7. The product of -1 and x is ________.
8. The product of -1 and -3 is ________.
9. The difference of $-x$ and $-x$ is ________.
10. The difference of 3 from 1 is ________.

The ten steps described verbally are the basic skills needed to perform the division example, $(x^2 - 4x + 1) \div (x - 3)$.

The teacher may obtain valuable information by first presenting the ten completion questions to the class and then following this set with the corresponding division problem. For example, some students may not answer all of the completion problems correctly but these students may solve the division example correctly. Except for possible careless errors, such students have difficulty in interpreting verbal statements algebraically.

Other students will answer the completion problems correctly, but these students will not divide correctly. These students have acquired the necessary skills but have not mastered the structure of the division process sufficiently to know when and where to perform these skills.

Some students may not be able to answer all the completion questions correctly. The incorrect answers to the completion questions may serve as a diagnostic test to point out just what skills are lacking.

Finally, this device will point out vividly to the student that the apparently complex division problems consist of a series of simple operations. For the student who has acquired the necessary skills, the mastery of the new process becomes a task of learning when and where to perform these skills in the new pattern. The

ladder may provide insight as to the true nature of the process for some students.

The following ladder illustrates a different type of sequence:

1. How many books at 5¢ can be bought with 75¢?
2. How many books at a half dollar can be bought with seven and a half dollars?
3. How many books at 25¢ can be bought with $3.25?
4. How many books at x cents can be bought with y cents?
5. How many books at n cents can be bought with $2n$ cents?
6. How many books at $2t$ cents can be bought with $6t^2n$ cents?
7. How many books at y cents can be bought with $(2y - 3y^2)$ cents?
8. How many books at $(x - y)$ cents can be bought with $(x^2 - y^2)$ cents?
9. How many books at $(x - 3)$ cents can be bought with $(x^2 - 2x - 3)$ cents?

In the solution of each of the above problems or steps, the second number is divided by the first number, but the difficulty of the division process changes in each succeeding example. Some students will solve the first several steps correctly but not get the correct order of division in the last step. Such a ladder will help to indicate the range of the student's mechanical ability to divide as well as to indicate to what extent he understands when to divide.

Diagnosis in Algebra

While algebraic ladders serve a definite diagnostic purpose, they are only partially a diagnostic instrument. There are times when a more thorough diagnostic test is required. The principles discussed in Chapter 13 should be kept in mind for algebra as well as arithmetic. Periodic quizzes given for the purpose of grading should also be diagnostic in a broad sense. The following diagnostic test indicates a comprehensive type of test on the solution of equations.[5] Other information on relative difficulty of items in algebra may be found on pages 44–45.

[5] Farnam, Laura M. Unpublished Masters Thesis, University of Minnesota, 1941.

Table XIII

Diagnostic Test on the Solution of Equations Given to Three Groups—X, Y, and Z

Test	Group X	Group Y	Group Z	Total
	Numbers below indicate the per cent of the group which missed the item in question			
I. a. $5a = 75$	2.9	4.7	0	3.2
b. $70 = 7a$	0	1.5	0	.8
c. $5x = 2$	5.8	4.7	14.2	7.2
d. $3 = 4a$	2.9	4.7	21.4	8.0
II. a. $3x = 12 + 3$	0	1.5	7.1	2.4
b. $5c = 29 - 4$	5.8	4.7	7.1	5.6
c. $23 + 12 = 7a$	2.9	1.5	3.5	3.4
d. $31 + 3 = 4b$	5.8	7.9	7.1	7.2
III. a. $c + 6 = 15$	8.8	9.5	7.1	8.8
b. $9 + d = 13$	0	6.3	14.2	6.4
c. $9 = c + 6$	2.9	4.7	7.1	4.8
d. $14 = 3 + c$	2.9	7.9	10.7	7.2
IV. a. $2a + 7 = 23$	0	1.5	7.1	2.4
b. $3 + 4c = 27$	2.9	3.1	0	2.4
c. $92 = 10d + 2$	2.9	3.1	3.5	3.2
d. $9 = 3 + 2t$	2.9	0	0	8.0
V. a. $7b + 2b = 36$	0	1.5	0	.8
b. $17c - 9c = 48$	0	6.3	0	3.2
c. $60 = 5b + 7b$	2.9	4.7	7.1	4.8
d. $30 = 11c - c$	2.9	3.1	0	2.4
VI. a. $3c + 4c + 2c = 36$	5.8	3.1	0	3.2
b. $48 = 9d - d + 4d$	0	1.5	3.5	1.6
c. $10b - 4b - b = 35$	2.9	4.7	3.5	4.0
d. $28 = c + 2c + 4c$	0	0	0	0

Test	Group X	Group Y	Group Z	Total
	Numbers below indicate the per cent of the group which missed the item in question.			
VII. a. $9a - 5a = 21 + 7$	8.8	4.7	7.1	6.4
b. $24 + 6 = 8a + 2a$	0	4.7	3.5	3.2
c. $14 - 6 = 3b + b$	0	1.5	10.7	3.2
d. $5c - 3c = 29 - 5$	5.8	3.1	14.2	6.4
VIII. a. $3a + 7a + 11 = 41$	0	6.3	3.5	4.0
b. $63 = 13a - 4a + 27$	2.9	7.9	7.1	6.4
IX. a. $5b + 3b + 7 = 39 + 8$	0	3.1	10.7	4.0
b. $65 - 13 = 13a - 4a + 2$	8.8	19.1	17.8	16.0
X. a. $5c + 9 + 2c - 7 = 46 - 9$	20.5	12.6	17.8	16.0
b. $19 + 15 = 6a + 9 - 3a + 4$	26.4	34.9	35.7	32.8

Table XIV

Summary of Errors in the Diagnostic Test on the Solution of Equations Given to Three Groups—X, Y, and Z

	Group X	Group Y	Group Z	Total
Total Number of Errors	47	121	71	239
Number of Students	34	63	28	125
Errors per Pupil	1.3	1.9	2.5	1.9

Group X has students with I.Q. range of 121–137
Group Y has students with I.Q. range of 104–120
Group Z has students with I.Q. range of 89–103

Summary

One of the major problems in the teaching of algebra is to maintain an effective balance between interpretation and manipulation. The manipulative phase of algebra may be made more meaningful by highlighting the principles of arithmetic that apply to the mechanics of algebra. The use of a set of principles, such as that on pages 434–436, allows the student to understand the orderly structure of the subject.

References at appropriate times to similarities and differences of algebra and arithmetic also may be of value. The reader may review these by referring to page 416.

The presentation of algebra as a language, a tool for generalization, and as a collection of puzzles will also help to maintain the balance between interpretation and manipulation. Emphasis should be placed on translation from algebraic to verbal language as well as from verbal to algebraic language. Whenever possible, the arithmetic background of an algebraic situation should be reviewed before the algebraic process is discussed. The application of algebraic relationships to particular numerical situations helps the student to interpret algebra arithmetically.

Verbal problems should include types requiring the development of a formula almost as often as types requiring conditional equations. The ability to analyze situations, recognize relationships, and express relationships algebraically is of great importance. Every effort should be made to develop these important skills.

The objectives and methods of teaching algebra discussed in this chapter will probably always be sound no matter what changes take place in the mathematics curriculum. Significant changes in the mathematics curriculum for college preparatory students are imminent. The College Entrance Examination Board has activated a commission to study the current curriculum and recommend changes in light of recent developments in mathematics. The Board intends to implement the findings of this commission by publishing source material on new developments in mathematics and to include questions on this material in future College Board Examinations. An alert teacher

dealing with students expecting to go to college will keep in touch with this program. It appears unlikely at the moment that this program will introduce any major changes in first year algebra. For a more detailed discussion of possible changes, see page 548.

Whether dealing with the traditional aspects of the curriculum, or with the new phases, group participation helps to clarify concepts. The new steps can be understood only when the ones from which they are derived have meaning to the learner. The students should be led to realize that a formula is a generalization of many specific problems, that it is a way of working all similar type problems in one operation.

Public Schools, Atlanta, Georgia

Questions, Problems, and Topics for Discussion

1. In what four ways should algebra be interpreted to students to help them to recognize that algebra is more than a series of manipulations?

2. In what four ways may a functional relationship be expressed in elementary algebra?

3. What two types of equalities are met in elementary algebra? How are these equalities related to the two common uses of letters for numbers in elementary algebra?

4. Give two points of similarity and two points of difference between algebra and arithmetic.

5. How would you help a student who makes these statements: "I never know when I am adding or subtracting. How can you add numbers by subtracting?"

6. What is the difference between an identity and a conditional equation?

7. What is the basic principle used in solving equations?

8. Why are first degree equations frequently called linear equations?

9. Match the following algebraic and verbal statements:

I. $0 \cdot x = 0$

II. $\frac{x}{-x} \equiv -1$

III. $xy \equiv yx$

IV. $ab \equiv a \div \frac{1}{b}$

V. $(a + b)^2 \equiv a^2 + 2ab + b^2$

VI. $3x - 2 = 7$

a. The square of the sum of any two numbers is equal to the square of the first number plus twice the product of the two numbers plus the square of the second number.

b. Any number multiplied by zero is zero.

c. 2 less than 3 times what number is 7?

d. The quotient of any number divided by its negative is -1.

e. To multiply by a number, divide by its reciprocal.

f. The order of the factors in a product does not affect the product.

10. Discuss two methods for solving equations with fractions.

11. Make a problem requiring the development of a formula which will be useful in some everyday situation. See page 432.

12. Construct a ladder of essential steps involved in the solution of the equation, $3\,(x - 5) = \frac{2}{3}x + 1$. See pages 450–451.

13. Discuss the relation between a proportion and an equation.

14. Find an article in a newspaper or magazine which makes use of a formula.

15. Give a verbal statement for the formula, $A = \frac{1}{2} h (a + b)$.

16. Outline the sequence of steps for presenting algebra as a tool for generalization.

17. Find an algebraic relationship with an interesting arithmetic interpretation.

18. Obtain a graph from a newspaper and prepare a series of questions which would test a person's ability to interpret this graph. Include questions which demand recognition of the function represented by the graph.

19. Analyze the steps in the diagnostic test on pages 452–453. Identify the new item of difficulty in each succeeding step.

Suggested Readings

Breslich, Ernst R. *Problems in Teaching Secondary-School Mathematics*, pp 119–151. Chicago: University of Chicago Press, 1931.

Butler, Charles H. and Wren, F. Lynwood *The Teaching of Secondary Mathematics*, pp. 268–338. New York. McGraw-Hill Book Co , 1941.

Davis, David R. *The Teaching of Mathematics*, pp. 193–212. Cambridge, Mass.: Addison Wesley Press, Inc., 1951.

Kinney, Lucien B. and Purdy, C. Richard *Teaching Mathematics in the Secondary School*, pp. 59–99. New York: Rinehart & Company, 1952

Reeve, William D. *Mathematics for the Secondary School*, pp 245–300 New York: Henry Holt and Co., 1954.

Chapter 13

Testing, Evaluation, and Diagnosis in Mathematics

In this chapter the following topics are discussed:

a. The nature and techniques of appraisal
b. Methods of appraisal applied to mathematics
c. Levels of diagnosis
d. Factors associated with lack of success in mathematics
e. The treatment of difficulties in mathematics.

a. The Nature and Techniques of Appraisal

The Purposes of Evaluation and Diagnosis

Evaluation is recognized as an educational task of crucial importance. The effectiveness of the mathematics program can best be measured by determining the growth of pupils toward desirable goals. These goals must be formulated before evaluation can be intelligently undertaken. On the basis of data concerning pupil growth and achievement intelligent decisions can be made by the staff concerning steps that should be taken to improve the curriculum, instruction, and materials and aids of learning and the organization of groups of students for instructional purposes.

A carefully planned evaluation program makes available unbiased information that helps the teacher and faculty to understand the individual pupil, his mental ability, his health,

his success in school, his personality traits, his interests, and his attitudes. Guidance in planning the total educational program of a particular student without adequate data concerning him is folly.

On the basis of information supplied by well constructed diagnostic tests in mathematics the teacher can make intelligent efforts to adapt instruction to the specific needs, the ability, and rate of growth of the individual learner. The pupil's awareness of his progress, strengths, weaknesses, difficulties, and needs as revealed by effective evaluation and diagnostic procedures serves as a powerful motivation of his learning activities. The most effective evaluation, from the standpoint of learning, is that which is done by the learner himself Evaluation should be regarded as an integral part of the teaching-learning program. Evaluation instruments should be available to teacher and learner whenever the learning situation requires them, not according to the calendar.

Steps in the Development of Means of Appraising Learnings

The sequence of steps in the development of means of appraising outcomes of any educational program, in this case mathematics, is generally considered to be as follows:

1. *Formulate the objectives clearly.*

The detailed analysis of the specific outcomes related to the technical and social phases of mathematics given on pages 2 and 3 provides an excellent basis for planning an evaluation program. To this list may be added other important outcomes of learning related to mathematics, including work habits and study skills, methods of thinking, skill in problem solving, ingenuity and resourcefulness in applying mathematical procedures, significant interests and attitudes, social sensitivity, and personal-social adjustment within the group. The formulation of objectives should be a cooperative enterprise, participated in by all who are concerned with the growth and development of the learner, including the student himself. The objectives that are recognized will affect all aspects of instruction.

2. *Clarify the meaning of the objectives.*

The meaning of each of the objectives must then be clarified by describing it in terms of student behavior and responses, which can be observed or tested to determine changes in the direction of the desired goals. When considering an available test, or in constructing one, questions such as the following should be considered: "Is the information or behavior required in this test related to an important objective of this course? Is the content of social value? Does the behavior called for give evidence as to the status of the objectives we have set up?"

3. *Collect test situations.*

Behavior should be evaluated in test situations that are broad enough to require responses by the learner comparable to those that he would make in functional situations. He should have the opportunity to exhibit one or more of the kinds of behavior listed in the second step. Problem solving situations are almost ideal for this purpose. Paper and pencil procedures are very useful as test situations for computational aspects of mathematics. Testing the ability of a student to measure some surface with a ruler requires observation of his performance in a specific situation. The test situations selected should be practicable from the standpoints of time, effort, and facilities required for their administration.

4. *Obtain a record of the student's behavior or responses.*

Paper and pencil procedures are a useful way of securing a record of responses on a test of computational skills. They may also be used to test knowledge of vocabulary and information about uses of mathematics. Standard tests and informal objective test exercises usually require use of paper and pencil techniques. In some situations the observation and recording of behavior, as when measuring an angle in indirect measurement, are necessary to obtain a record. Anecdotal records also are useful when the teacher, parents, or students wish to make a report of some significant behavior or response by a particular individual or group. Questionnaires and interviews can be used to secure

helpful information about the student's methods of work, his interests, and his attitudes toward mathematics. Whatever the form of record, it should describe accurately all of the significant reactions of the student that may be of value in interpreting the report. The larger the number of significant reactions in a variety of situations, the more dependably and objectively the behavior can be evaluated.

5. *Interpret and evaluate the record.*

Behavior should be evaluated in terms of the important objectives that have been established and in terms of the characteristics of the individual student.

Norms for standard tests make it possible for a teacher to compare the achievement of a particular pupil with the average performance of other pupils. Due consideration should be given to the individual's capacity to learn, background, and maturity in interpreting his record. The purpose of the teacher should be to "take the pupil from where he is to where he ought to be." Although information as to his present status with regard to a particular outcome is of undoubted value, special attention should be given to determine the progress that the student is making toward the achievement of commonly accepted goals of learning.

In fields for which objective standards are available, as in the measurement of scholastic aptitude, the interpretation of the data is relatively simple. The evaluation of achievement in mathematics is much more difficult because the desirable outcomes of instruction in this field are numerous, many sided, less definite, and in some cases, such as interests and attitudes, not susceptible to precise measurement. For some outcomes, for instance, attitudes toward specific areas of mathematics, no suitable methods of appraisal have as yet been devised.

Aspects of Intellect to be Considered in Evaluation and Diagnosis

In evaluating the performance of a student on a test or in his daily work, five aspects of intellect should be considered, namely, (1) rate of work, (2) accuracy, (3) level of difficulty of items to

which correct responses were made, (4) area of ability, and (5) methods of work.

1. *Rate of work.* The speed with which an individual responds is an index of his control over the elements or skills being tested, assuming that his responses are accurate, intelligent, and meaningful.

2. *Accuracy.* The number or the per cent of items answered correctly is ordinarily a useful measure of accuracy. Other things being equal we regard the student who responds quickly and correctly as performing satisfactorily.

3. *Level.* The higher the level of difficulty of the items to which the student can respond correctly, the more highly his ability is rated.

4. *Area of ability.* The wider the range of correct responses the individual student makes to items of the same level of difficulty, the broader is the area of his intellect. Attitudes, interests, and appreciations are considered in this concept.

5. *Methods of work.* The more efficient, economical, and intelligent the student's methods of work are, the higher we tend to rate his performance. The means used by the learner to arrive at the answer are as important from the standpoint of evaluation as the end product itself.

Standard tests in mathematics usually measure level of ability, with the criterion of accuracy a key factor. It is very difficult to measure rate of work directly because of the complex nature of mathematical skills, but we can observe the pupil at work and note his speed of response as compared with that of others. Area of mathematical ability is closely related to the varied outcomes desired in this field. When tests are limited to computational skills and instruction stresses these relatively narrow outcomes, other significant areas of learning are certain to be neglected. Methods of work are usually not measured directly by paper and pencil tests, but can be evaluated by an observer on the basis of the observation of the performance of the student in some test or learning situation. Diagnosis in mathematics is particularly concerned with the methods of work of the student, his thought processes, and the resourcefulness and interest with which he approaches the task at hand.

Public Schools Portland Oregon

The use of different types of exploratory materials is effective in evaluation and diagnosis.

b. Methods of Appraisal Applied to Mathematics

The outline below contains an analysis of the wide variety of objective tests and evaluative procedures that are used in the appraisal of outcomes in mathematics. In the sections that follow various applied procedures are discussed.

I. Standard tests and objective procedures
   1. Standard tests
      a. Achievement tests
      b. Readiness tests
      c. Prognostic or aptitude tests
      d. Diagnostic tests in specific phases
      e. Attitude scales
   2. Unstandardized short answer objective tests
      a. Simple recall or free response
      b. Completion
      c. Alternate response
      d. Multiple choice
      e. Matching
   3. Improved essay type examination
   4. Scales for rating a product or a performance

II. Evaluation of learning and behavior by less formal procedures
   1. Problem situation tests—actual experience, or indirect approach
   2. Behavior record
      a. Controlled conditions involving check lists, rating scales, time studies, recordings, etc.
      b. Uncontrolled conditions, involving anecdotal records, diaries, reports, observations of behavior and responses in classroom and elsewhere
   3. Inventories and questionnaires about attitudes, interests, activities, methods of study
   4. Interviews, conferences, personal reports
   5. Analysis of elements of some product
   6. Sociometric procedures

Standardized Survey Tests

The purpose of survey tests is to provide a basis for determining the effectiveness of the mathematics program as a whole, as measured by the progress made from year to year by students at all levels of the school toward the achievement of accepted objectives. The data supplied by a dependable survey test also give the teacher information about the individual pupils on the basis of which to plan the instructional program. Some of the widely used tests in mathematics for junior high schools including arithmetic, general mathematics, and algebra are listed below.

Standardized Survey Tests in Arithmetic and General Mathematics

1. Analytical Scales of Attainment in Arithmetic, Division 3 (Educational Test Bureau)
2. California Achievement Test (California Test Bureau)
3. Coordinated Scales of Attainment, Batteries 6, 7, 8 (Educational Test Bureau)
4. Davis Test of Functional Competence in Mathematics
5. Iowa Every Pupil Tests of Basic Skills, Test D, Advanced examination. Basic Arithmetic Skills (State University of Iowa)
6. Iowa Tests of Educational Development–Test 4–Ability to do Quantitative Thinking (Gaines Research Associates)
7. Lankton First-Year Algebra Test (World)
8. Larson-Greene Unit Tests in First Year Algebra (World) Series of 6 Unit Tests
9. Metropolitan Achievement Test
10. New Stanford Achievement Examination (World)
11. Seattle Algebra Test (World)
12. Snader General Mathematics Test (World)
13. Sueltz Functional Evaluation in Mathematics–Upper Level (Educational Test Bureau)

Aptitude and Prognostic Tests

The function of aptitude and prognostic tests in algebra and geometry is to supply basic data to be used in grouping students

according to their likelihood of success in these fields and for general guidance. There are two approaches to prognosis in these areas (1) tests of the speed and accuracy with which the student is able to acquire skills and information in each field, characteristic of the Orleans prognostic tests in algebra and geometry, and (2) tests which inventory underlying skills and abilities previously learned on which success in the new subject depends.

Aptitude and Prognostic Tests

(1) Tests of ability to learn skills and information
 Orleans Algebra Prognosis Test (World)
 Orleans Geometry Prognosis Test (World)

(2) Inventory tests of underlying skills necessary for success
 California Algebra Aptitude Test (California Test Bureau)
 Iowa Algebra Aptitude Test (Revised) (University of Iowa)
 Iowa Placement Examination in Mathematics Aptitude (University of Iowa)
 Iowa Plane Geometry Aptitude Test (University of Iowa)
 Lee Test of Algebraic Aptitude (World)

The most significant uses of prognostic tests are (1) discovering students with unusual mathematical aptitude and ability, (2) determining fitness of students for particular courses or programs, and (3) aids in classifying and grouping students.

In general the order of merit of various bases for predicting mathematical achievement is (1) ratings on dependable prognostic tests, (2) ratings of mental ability, (3) scores on comprehensive achievement tests on previous work in mathematics, (4) average mark in previous years. A practical basis of grouping is a combination of mental ability ratings and average mark in previous courses.

An illustration of the second type of test is the Iowa Algebra Aptitude Test (Revised) which contains tests of factors, none of which involve algebra but are known to be highly related to achievement in algebra, namely, (1) arithmetic computations, (2) computations involving abstract concepts, (3) manipulation of numerical series, and (4) solution of verbal problems involving dependence and variation. Pupils whose scores on the test fall

below the twentieth percentile of standard scores are almost certain to fail in algebra, or to be so weak that they will require special attention by the teacher to achieve even a minimum of success. In many schools such students are guided into less difficult courses in basic mathematics or general mathematics, or courses in applied mathematics.

Objective and Short Answer Tests

The items below illustrate various types of objective and short answer exercises that can be used in preparing tests of the various outcomes and abilities indicated under each of the types:

I. Simple recall or free response items

a. Vocabulary

1. In the fraction $\frac{3}{4}$ what do we call the 3? ________

b. Computational skill

1. How much is 6×0? _____

c. Understanding

1. Express .05 as a per cent.

II. Completion items

a. Perception of relations

1. $2 + 4 = 9 - ?$
2. $4 \times ? = 6 + 4$

b. Recognition of generalizations

1. When we subtract 1 from a whole number, the answer is the next ________ number.

c. Understanding

1. When we divide a proper fraction by a larger proper fraction, the quotient is less than _____.

III. Alternate response items

a. Recognition of generalizations

1. Does $2 \times 235 = 235 \times 2$? Yes—No

b. Understanding of place value

1. In the number 36.094 is 0 written in ones' place or in tenths' place? ________

c. Estimating answers

1. The product of 4.75×2.6 is more than 10. True—False

IV. Multiple choice items

a. Vocabulary

1. What do we call the money we pay for protection or insurance?

a. premium b. dividend c. discount d. commission

b. Ability to select solution of problem

1. Find the perimeter of a garden 60 feet long and 40 feet wide.

a. Add 60 and 40.

b. Find 60×40.

c. Find the sum of 2×60 and 2×40.

d. Find $\frac{1}{2}$ of $60 + 40 + 60 + 40$.

c. Estimating answers to check solutions

1. The quotient of $278\overline{)19{,}784}$ is nearest:

a. 900   b. 600   c. 70   d. 90

d. Evaluating sizes of numbers

1. Which of the following numbers is largest in value?

a. 3.2   b. 3.1478   c. 3.19   d. 3.204

V. Matching items

Vocabulary

Draw a line connecting each word in A with its matching idea in B.

A	B
1. Reciprocals	Two equal ratios
2. Algorism	$\frac{3}{4}, \frac{4}{3}$
3. Proportion	$A = \frac{1}{2}ab$
4. Formula	$3\overline{)36}$ with quotient 12

Applying Appraisal Procedures

The technique of evaluation selected should be appropriate for the purpose and yield dependable information about the status of the outcomes being appraised. The following analysis lists procedures for evaluating a number of important areas of learning in mathematics.

Outcomes	Techniques to Apply
1. Computational skills Understanding of operations Ability to apply Perception of relationships	Standard tests showing growth made Informal objective tests of facts and processes Interview of pupil to check on understanding Observation of daily work Anecdotal records of ability to apply processes in social situations
2. Problem solving In life situations Textbook problems Described situations	Standard tests Informal objective tests Analysis of daily work Observation of behavior in dealing with problematic situations Anecdotal records of insight into social significance of mathematics
3. Construction and interpretation of graphic and tabular materials	Standard tests of study skills Informal objective tests Oral discussion of graphs, tables, etc Note instances of interpretation in other classes Evaluate some graph or table prepared by the pupil
4. Use of measurements Units of measure, Precision instruments	Test of knowledge of units of measure Observe and rate skill of manipulating and using measuring devices in test situations and in free actions Record evidences (activities, comments, etc.) of resourcefulness in using measures
5. Interests and appreciation	Interest inventory Records of interests and attitudes revealed by behavior, responses, contributions Pupil ranking of mathematics among various courses taken

Outcomes (*cont.*)	Techniques to Apply (*cont.*)
	Records of enrollment in later courses Ratings by pupils of interest in various topics
6. Methods of work and study	Observation Interview Essay

c. Levels of Diagnosis

Diagnosis may be conducted at several different levels. The procedure to be used will be determined by the depth to which it is desired or necessary to carry the diagnosis. The first level may be called *general* diagnosis. At this level standard survey tests are used to obtain a measure of general ability in mathematics. These tests usually reveal (1) a wide range of abilities among the different pupils of a group, and also (2) a wide variation in the development of specific areas of ability in the case of an individual student, shown by an uneven profile of test scores.

The second level may be named *analytical* diagnosis. Here the teacher administers to a class as a whole diagnostic tests that are so constructed that the tests *locate* specific weaknesses in any particular area of skill, such as addition of common fractions, or division by two-place numbers. Remedial steps can then be taken to correct the difficulty.

At the third level the approach to diagnosis is on an individual or *case-study basis*. This is necessary when the complex nature of learning difficulties of some pupil requires systematic study or must be approached on a clinical basis to determine the nature and causes of inferior performance. At this level the teacher may use informal tests of specific skills constructed on the spot as the need arises, or apply more systematic standardized diagnostic instruments that are available which enable the teacher to study the characteristics of the pupils' thought processes and methods of work to determine the exact nature of his difficulty.

Steps in the Detailed Diagnosis and Treatment of Learning Difficulties

A generalized approach to diagnosis of learning difficulties in mathematics by either teacher or clinical worker includes the following steps:

1. The setting up of educational objectives as guides for both teacher and learner.
2. The appraisal of the educational product by evaluative procedures such as those discussed earlier in this chapter to determine the strengths and weaknesses of the students in relation to the various desirable outcomes.
3. The recalling and informal consideration of possible factors in the teaching-learning situation that may be conditioning the growth of the learner unfavorably Here both previous experience and the findings of scientific research should be reviewed for ideas as to what the factors may be that underlie the disability.
4. A preliminary survey of the particular situation to identify possible factors that may be operative in causing unsatisfactory progress in the case under consideration.
5. The formulation of ideas as to (a) factors that should be systematically analyzed, (b) factors that require merely routine study, and (c) factors that are likely not to be operative and hence need not be investigated.
6. The use of systematic analytical procedures to study the situation so as to establish with some assurance the existence and significance of the factors that it is thought may be contributory.
7. The planning and carrying out on a tentative basis of what is believed to be a more effective developmental program including constructive changes in any or all aspects of the total teaching-learning situation that may seem advisable.
8. The subsequent re-evaluation of the changes in the behavior of the learner to establish the validity of the diagnosis, the effectiveness of remedial measures, and the basis of further guidance.

The above analysis outlines the procedures that a skillful teacher or diagnostician follows in making a case study. With experience in diagnosis the steps merge and are not necessarily followed

in 1-2-3 order as listed. The authors wish to make the point that the teacher should systematically study the work of any pupil who is failing to progress satisfactorily to determine the nature of his difficulty, and its causes. Steps can then be taken to do any necessary reteaching, and in cases of serious disability readjustments can be made of curriculum, methods of instruction, and learning materials that are believed to be most likely to lead to improvement.

An Illustration of General Diagnosis

The data in Table XV present the results of the Iowa Test of Basic Skills in Arithmetic for a typical seventh grade class. They illustrate the nature of the results of any of the general survey tests listed on page 465.

The median scores for each part of the test were slightly below the norms for the grade. The greatest difference between the median and the grade norm was in problem solving. The median mental grade level of the class was slightly below the norm for the grade, about 3 months. The apparent general deficiency in problem solving requires further investigation.

TABLE XV

DISTRIBUTION OF SCORES OF A GRADE 7 CLASS ON THE IOWA TEST OF BASIC SKILLS IN ARITHMETIC

Grade	Vocabulary	Number Operations	Problem Solving	Mental Grade
10.0	1	—	—	2
9.0	2	4	5	2
8.0	3	9	4	4
7.0	12	10	7	9
6.0	9	7	11	8
5.0	4	3	6	4
4.0	3	2	2	1
3.0	1	—	—	—
Median	7.1	7.5	6.9	7.3
Norm	7.6	7.6	7.6	7.6

Number of cases—35

The most striking point about the data is the extremely wide variation in the scores of individual students on all parts of the test and in mental ability. In vocabulary the range was approximately 7 years, in number operations, 5 years, and in problem solving, 5 years. At a glance it can be seen that 8 students were more than $1\frac{1}{2}$ years below the norm in vocabulary; 5 students were similarly retarded in number operations, and 8 in problem solving.

It is necessary to take into consideration the student's mental level in evaluating his performance on the test. Consider the results for three of these students given below:

	I.Q.	*Vocabulary*	*Operations*	*Problems*	*Mental Grade*
Tom	90	6.4	7.2	6.5	6.1
Mary	104	7.8	7.8	6.6	7.9
Arthur	140	8.4	8.0	9.1	9.4

Tom is somewhat below normal in mental ability. However his scores are all higher than his mental grade although they are below the norm for his grade. Tom thus is exceeding the level of expectancy of students of his intellectual capacity. Arthur on the other hand has a high mental rating but his scores are all below his level of expectancy. Perhaps he has not been challenged by the work in mathematics. Mary's performance on the whole is approximately what would be expected except in problem solving where her performance is more than one year retarded.

The unevenness of the scores on the three parts of the test is characteristic of the results for individual students in any class. The data supply valuable information about each individual on the basis of which the teacher can take steps to adjust the instructional program to the needs of each individual. However, more detailed study is necessary of the work of particular students whose performance in any of these areas is regarded as unsatisfactory.

The reader may speculate as to the "marks," if any, each of these three students should receive.

1. *Normal or above normal progress*

The performance of these students is in line with levels of expectancy. For them the regular program is satisfactory, although it is desirable to strengthen the instruction to secure even better results.

2. *Simple retardation*

These students are performing at a level somewhat below expectancy but they have no apparent disability requiring special treatment. They often lack necessary experience and background, but their work can easily be improved by careful guidance.

3. *Specific disability cases*

These students have specific weaknesses that interfere with successful growth. For instance, an apparent disability in division by two-place numbers may be due to a deficiency in subtraction which can be remedied by systematic diagnosis of difficulties and reteaching of the process as may be necessary. Performance in operations may be at a low level because of lack of knowledge of basic number facts which causes the student to count in a variety of ways or to guess when giving answers. Treatment of such cases can usually be undertaken by the classroom teacher.

4. *Complex disability cases*

These students for a variety of reasons have made little if any progress in mathematics. They have acquired a dislike for the subject because of inability to learn. They fear the subject and become emotionally upset when working on it. They often do not understand the work and have serious deficiencies in underlying skills for a variety of reasons. Often they are normal or above in mental ability. They present such serious problems that often the services of specialists are necessary to make the diagnosis and to assist the teacher to plan a remedial program. Unless a dependable diagnosis is first made, treatment cannot be effective. In cases of extreme disability, treatment on a clinical basis may be necessary.[1]

[1] For a comprehensive discussion of case-study procedures, the reader should read the volume, Brueckner, L. J. and Bond, Guy L. *The Diagnosis and Treatment of Learning Difficulties.* New York: Appleton-Century-Crofts, Inc., 1955.

Techniques of Analytical Diagnosis

The purpose of analytical diagnosis in mathematics is to determine more definitely the specific deficiencies and areas of difficulty within number processes, problem solving, and other phases of the subject.

The diagnostic procedure to be applied in a particular situation depends on the purpose of the teacher. Certain kinds of diagnostic tests are designed chiefly for use with groups of students; others are intended for clinical diagnosis on a case-study basis. We shall discuss three kinds of analytical diagnostic tests, (1) inventory or screening tests, (2) tests of knowledge of basic number facts, and (3) analytical diagnostic tests used to locate specific weaknesses in some process. The nature of diagnostic tests for use in individual case studies will be discussed in a later section.

1. *Inventory tests at beginning of year's work or as needed*

Inventory tests are used to get a quick overview of the ability of one or more pupils in some basic mathematical process or topic such as the use of per cents. The diagnostic test in per cent given on pages 246–247 is intended for use in Grades 8 and 9 The test is divided into five parts, each of which consists of a number of items that sample a range of skills in each of the areas involved. When administered by the teacher at the time when the review of this topic begins early in the year in Grade 8 or 9, a scrutiny of the results of the whole test not only gives the teacher information as to the variation in ability in this field among the students in a class, but also an indication as to the specific aspects of the topic in which the various pupils have difficulty. This is revealed by the number of incorrect answers in each of the five parts of the test. A further analysis of the errors by pupils whose test scores were unsatisfactory on the whole or part of the test will reveal to the teacher the needs of the pupils and thus make it possible to plan an effective remedial program.

Inventory tests of this kind are often included in textbooks. When they are not provided, the teacher can quite easily prepare them by following the model of the screening test in per cent.

Clinics often use screening tests similar in form to the inventory test on page 132 when cases are referred to them for diagnosis. They help the examiner to locate in a short time general areas of strength and weakness so that the detailed diagnosis can proceed.

2. *Testing knowledge of number facts*

The single most common cause of incorrect work in arithmetic processes is lack of mastery of the basic number facts. A quick way to test the pupils' knowledge of the basic facts in some process is to administer a test of the 50 most difficult facts in that process according to either of the plans described below and then to have each student do special work on the facts in each process that he finds difficult. Some students who are especially weak may need to study most of the facts in one or more processes to increase their rate and accuracy of work.

PLAN I

1. Prepare a test of the most difficult facts in an operation, similar to the test for addition on page 477.

2. Give each pupil a sheet of paper containing rows of 1 inch squares (mimeographed or on ditto paper).

3. Dictate the facts one at a time. After the first fact has been dictated, count "one-two" and then dictate the next fact. Continue at this rate until all facts have been dictated.

4. To score the papers have students exchange papers and dictate the answers; or distribute copies of the test paper containing the facts and answers. Have the students mark each answer: *C—correct, X—incorrect, or O—omitted. Have each* student find the number of correct answers. Have each student make a list of the facts he either omitted or answered incorrectly, indicating that they were not known.

5. Plan special work for pupils whose test papers in any process reveal the need of extra study and practice.

PLAN II

1. Prepare a test of the most difficult facts in an operation, similar to the test on page 477.

2. Distribute copies of the test, one for each pupil.

3. As you dictate the facts, the students see them on the test paper. However, to prevent counting and to control rate of work, dictate the facts at a set rate as described above under (3) for Plan I. When a pupil cannot write an answer in the time allowed, he should proceed to the next fact dictated by the teacher.

4. The scoring of the test and the use of the test results are the same as for Plan I.

TEST OF ADDITION FACTS

	a	b	c	d	e	f	g	h	i	j
1.	7 6	8 9	5 9	9 9	6 8	5 5	9 4	7 4	5 8	7 9
2.	6 9	9 3	8 2	7 3	8 6	6 5	2 9	8 3	4 9	9 6
3.	9 8	7 5	4 8	7 8	9 1	4 7	8 8	6 7	3 9	6 4
4.	6 6	5 4	9 5	5 7	8 5	3 7	4 5	5 6	2 8	2 7
5.	8 7	7 7	3 8	9 2	4 6	6 3	9 7	7 2	3 6	8 4

3. *Analytical tests of component skills*

Some diagnostic tests are constructed in such a way that they reveal points at which a student's knowledge of a major skill, such as addition of fractions, tends to break down or reveal the component elements of a major skill such as division by two-place numbers in which there may be a weakness that is causing the major difficulty.

The diagnostic test in addition of fractions on page 478 illustrates the first type of diagnostic test. In Part I, row 1 contains examples in which there is no reduction of the sum of two fractions; row 2 requires reduction of the sum; in row 3 the sum is a

mixed number not requiring reduction; in row 4 the sum is a mixed number requiring reduction. In Part II of the test the analysis of skills is similar, except that the sums, in all cases, involve improper fractions that must be expressed in simplest form. In rows 5 and 6 the improper fractions are reducible to 1. Differentiated sets of examples can be made for fractions in the same family, as $\frac{1}{2} + \frac{1}{4}$; for fractions in which a common denominator is not present, as $\frac{1}{2} + \frac{1}{3}$; and for fractions in which the common denominator has a common factor, as in $\frac{1}{4} + \frac{1}{6}$. A single error in a set of three examples can be regarded as a chance error, readily corrected. When there is more than one incorrect answer in a particular set of three examples in the test, a persistent difficulty

Analytical Diagnostic Test in Addition of Like Fractions

Part I (Sums with Proper Fractions)

	a	b	c		a	b	c
1.	$\frac{1}{3}$ $\frac{1}{3}$	$\frac{1}{5}$ $\frac{2}{5}$	$\frac{3}{6}$ $\frac{2}{6}$	3.	$1\frac{1}{3}$ $2\frac{1}{3}$	$3\frac{2}{5}$ $1\frac{2}{5}$	$4\frac{1}{6}$ $3\frac{5}{6}$
2.	$\frac{1}{4}$ $\frac{1}{4}$	$\frac{1}{8}$ $\frac{3}{8}$	$\frac{3}{10}$ $\frac{3}{10}$	4.	$3\frac{1}{4}$ $2\frac{1}{4}$	$2\frac{3}{8}$ $1\frac{1}{8}$	$4\frac{3}{10}$ $3\frac{5}{10}$

Part II (Sums with Improper Fractions)

	a	b	c		a	b	c
1.	$\frac{2}{3}$ $\frac{2}{3}$	$\frac{3}{5}$ $\frac{4}{5}$	$\frac{7}{8}$ $\frac{6}{8}$	4.	$3\frac{3}{4}$ $2\frac{3}{4}$	$5\frac{7}{8}$ $\frac{3}{8}$	$7\frac{5}{6}$ $3\frac{5}{6}$
2.	$\frac{3}{4}$ $\frac{3}{4}$	$\frac{5}{8}$ $\frac{7}{8}$	$\frac{5}{6}$ $\frac{5}{6}$	5.	$\frac{1}{3}$ $\frac{2}{3}$	$\frac{1}{2}$ $\frac{1}{2}$	$\frac{3}{8}$ $\frac{5}{8}$
3.	$4\frac{2}{3}$ $6\frac{2}{3}$	$\frac{3}{5}$ $2\frac{3}{5}$	$4\frac{4}{6}$ $2\frac{3}{6}$	6.	$5\frac{2}{5}$ $6\frac{3}{5}$	$4\frac{1}{4}$ $6\frac{3}{4}$	$7\frac{5}{6}$ $\frac{1}{6}$

Note: This diagnostic test is more detailed than the test in fractions on page 176. It contains three examples of each step type to insure reliable diagnosis in disability cases.

Diagnostic Tests and Self-Helps in Arithmetic

DEVISED BY LEO J. BRUECKNER

TEST NO 10A

Diagnostic Test in Division by Two-Place Numbers

Name ______________ Grade or Course ________ Age ____

School ____________ Teacher ________ Room ____ Date ______

I. Finding Quotients

	a	b	c	d	e
1.	$10\overline{)60}$	$20\overline{)80}$	$40\overline{)240}$	$60\overline{)300}$	$90\overline{)630}$
2.	$10\overline{)43}$	$20\overline{)69}$	$30\overline{)86}$	$50\overline{)358}$	$70\overline{)528}$
3.	$44\overline{)89}$	$24\overline{)75}$	$51\overline{)306}$	$85\overline{)599}$	$79\overline{)402}$
4.	$24\overline{)81}$	$37\overline{)211}$	$43\overline{)412}$	$58\overline{)511}$	$16\overline{)110}$

No. Correct I

1. ____
2. ____
3. ____
4. ____

Total ____

II. Multiplication and Subtraction as Used in Division

Complete the examples. The quotients are correct

	a	b	c	d	e	f
1.	$\begin{array}{r}3\\48\overline{)162}\end{array}$	$\begin{array}{r}5\\67\overline{)349}\end{array}$	$\begin{array}{r}9\\94\overline{)878}\end{array}$	$\begin{array}{r}7\\60\overline{)479}\end{array}$	$\begin{array}{r}4\\89\overline{)372}\end{array}$	$\begin{array}{r}7\\17\overline{)128}\end{array}$
2.	$\begin{array}{r}2\\56\overline{)126}\end{array}$	$\begin{array}{r}6\\86\overline{)534}\end{array}$	$\begin{array}{r}9\\76\overline{)690}\end{array}$	$\begin{array}{r}3\\57\overline{)183}\end{array}$	$\begin{array}{r}7\\84\overline{)631}\end{array}$	$\begin{array}{r}8\\39\overline{)319}\end{array}$
3.	$\begin{array}{r}6\\59\overline{)403}\end{array}$	$\begin{array}{r}4\\24\overline{)211}\end{array}$	$\begin{array}{r}5\\19\overline{)106}\end{array}$	$\begin{array}{r}8\\48\overline{)391}\end{array}$	$\begin{array}{r}2\\87\overline{)186}\end{array}$	$\begin{array}{r}9\\18\overline{)164}\end{array}$

No. Correct II

1. ____
2. ____
3. ____

Total ____

Related Tests 1, 2, 3, 4, 5, 6, 7, 8, & 9

Self-Helps in Division by Two-Place Numbers

1. Study the work of one row of examples at a time to see that the quotients and remainders, if any, are correct. Then copy the examples on scratch paper without the answers and divide. Check your answers. Then see if your answers are correct.

I. One-Place Quotients

Lines 1-2: Dividing by even tens
Line 3: Dividing by other two-place numbers; no correction necessary
Line 4: Dividing by two-place numbers; correction necessary

	a	b	c	d	e
1.	6 10)60 60	4 20)80 80	6 40)240 240	5 60)300 300	7 90)630 630
2.	4 10)43 40 3	3 20)69 60 9	2 30)86 60 26	7 50)358 350 8	7 70)528 490 38
3.	2 44)89 88 1	3 24)75 72 3	6 51)306 306	7 85)599 595 4	5 79)402 395 7
4.	3 24)81 72 9	5 37)211 185 26	9 43)412 387 25	8 58)511 464 47	6 16)110 96 14

For Extra Practice (Copy and divide)

1.	20)60	10)70	40)320	70)560	90)810
2.	10)56	20)48	30)75	60)367	80)702
3.	43)86	21)59	32)128	85)599	76)617
4.	87)600	59)402	48)263	39)318	17)128

See Test 10B for Test in Dividing by Two-Place Numbers (Two- or More Place Quotients)

is indicated which should be diagnosed and corrected.[2] Some arithmetic textbooks contain excellent diagnostic tests of this type for all number operations.[3]

The second type of analytical diagnostic test is illustrated by the diagnostic test in division by two-place numbers on page 479. This process is in fact a complex of the four basic processes, as can be seen by an analysis of the steps in solving the division example at the right: First, simple division ($3\overline{)20}$) is used to find the quotient figure; next, the student must multiply 34 by 6, a procedure that involves addition by endings in carrying; and finally, he must subtract to find the remainder.

$$\begin{array}{r} 6 \text{ r}3 \\ 34\overline{)207} \\ \underline{204} \\ 3 \end{array}$$

The diagnostic test in division on page 479 is one of a series of 23 analytical diagnostic tests in arithmetic devised by Brueckner which enables the teacher to determine in which of three component skills there may be the weakness that is causing incorrect work in division with two-place numbers. Test 10A Part I tests the ability of the pupil to estimate the quotient figure, with and without correction; Part II tests the ability to multiply and subtract in the form used in division. Test 10B contains a systematic arrangement of division examples with quotients of two or more figures, including zero difficulties in quotients, and all difficulties of estimation of quotient figures.

The names of the series of analytical diagnostic tests and the test numbers are listed below.[4]

	Test Numbers
1. Tests of the basic facts (5 tests)	1
a. Test in Addition Facts (2 Sections)	2
b. Test in Subtraction Facts (2 Sections)	3
c. Test in Multiplication Facts (2 Sections)	

[2] Brueckner, L. J., and Elwell, Mary "Reliability of Diagnosis in Multiplication of Fractions," *Journal of Educational Research* 26.175–185
Brueckner, L. J, and Hawkinson, Ella "The Optimum Order of Arrangement of Items in a Diagnostic Test," *Elementary School Journal* 34.351–357.

[3] *Winston Arithmetics* for Grades 7 and 8.

[4] The tests are published by the California Test Bureau, Los Angeles, California (1955).

	Test Numbers
d. Test in Division Facts (2 Sections)	4
e. Test in Uneven Division Facts (2 Sections)	5
2. Tests in the four operations with whole numbers (5 tests)	
a. Addition (3 Sections)	6
b. Subtraction (4 Sections)	7
c. Multiplication (2 Sections)	9
d. Division by one-place numbers (3 Sections)	8
e. Division by two-place numbers (3 Sections)	10A, 10B
3. Tests in common fractions (7 tests)	
a. Changing the form of fractions (3 Sections)	11
b. Addition	
(1) Like fractions (2 Sections)	12
(2) Unlike fractions (2 Sections)	14
c. Subtraction	
(1) Like fractions (2 Sections)	13
(2) Unlike fractions (2 Sections)	15
d. Multiplication (2 Sections)	16
e. Division (2 Sections)	17
4. Tests in decimal fractions (4 tests)	
a. Addition (4 Sections)	18
b. Subtraction (5 Sections)	19
c. Multiplication (6 Sections)	20
d. Division (3 Sections)	21
5. Test in Per Cent (6 Sections)	22
6. Test in Operations with Measures (5 Sections)	23

Each diagnostic test has cross-references to "Self-Helps" to be used for any necessary remedial work. These "Self-Help" materials appear on the back page of each test. This combination of diagnostic tests and self helps provides the essential elements of an effective improvement program in arithmetic operations.

Relationship between General Ability and Component Skills

As has been shown, number operations are composed of various component elements which the student must have under

control in order to be successful. To determine the relationship between ability to divide by two-place numbers and a number of the component skills of the operation, Wiest[6] administered a general test in division to over 200 Grade 6 pupils and also separate tests of each of the components. Then he found the correlation between the major test scores and the scores on the tests of the component skills with the results given below.

Correlation of Subtest Scores and Scores on General Tests in Division by Two-place Numbers

1. Division by one-place numbers	.79
2. Estimating first quotient figure	.74
3. Multiplication as in division	.67
4. Vocabulary of division	.62
5. Finding errors in division	.60
6. Subtraction	.48
7. Judging correctness of given quotients	.46
8. Placing the first quotient figure	.43

Note the large differences in the relationships of the components and the general ability to divide. The highest correlations are with ability to divide by one-place numbers and with ability to estimate the first quotient figure. The ability to multiply as in division was much more highly related to the general ability to divide than was subtraction. Judging the correctness of given quotients and placing the first quotient figure correctly correlated least with general ability to divide. Wiest also found that in general good achievers scored above the 50 percentile of the group scores on all subtests, while many of the poor achievers tended to score below the 50 percentile on all of the tests. The striking fact was discovered that in a considerable number of cases poor achievers scored low on one or two of the subtests but fairly high on the others. This indicates that in some cases disability in division may be due to a specific deficiency in one or more of the component skills such as subtraction, a fact of considerable significance in relation to diagnosis. It is likely that

[6] Unpublished study, William Wiest, University of Minnesota graduate student.

steps taken to improve weakness in a component skill would under such circumstances produce a general improvement in division.

The Compass Diagnostic Test in Problem Solving illustrates the possibility of analytical diagnosis in problem solving. The test consists of five parts, each testing a different component, as follows:

PART I—Comprehension (a reading test)
PART II—What is given?
PART III—What is called for?
PART IV—Probable answer (estimation)
PART V—The correct solution (a choice of five)

Arithmetic textbooks and workbooks[6] often provide exercises that are suitable for diagnosis in problem solving.

Case-Study Procedures

Techniques used in case studies may be grouped under two headings, (1) standardized tests for individual diagnosis, and (2) informal diagnostic procedures.

1. *Standardized clinical diagnostic tests*

Several diagnostic tests are available that are especially intended for the study of the work of pupils whose work in arithmetic is seriously deficient. These tests include a systematically constructed series of subtests ranging from the simplest combination of skills involved in a process, for instance, addition of whole numbers, to as complicated an example in that process as students are expected to learn to work in school.

This kind of diagnostic test is given individually by the examiner who records on a special blank that is provided with the test the errors, faulty procedures, and other symptoms of difficulty revealed in the course of the examination. The basis of this analysis is the findings of systematic research as to the kinds of difficulties most frequently found in the work of pupils. This information gives the teacher clues as to what to look for in making a diagnostic study.

[6] See the series of workbooks *Guided Study and Practice in Arithmetic* for Grades 7 and 8 that are published by The John C. Winston Co.

For illustrative purposes the list of the most frequent errors in operations with decimals made by a group of 168 seventh grade students is given below:

	Per Cents of Total Number
1. Addition of decimals	
a. Misplacing decimal point in the sum	47.4
b. Computational errors as such	27.4
c. Errors in adding common and decimal fractions	7.4
Number of errors analyzed–580	
2. Subtraction of decimals	
a. Computational errors as such	73.8
b. Misplacing decimal number in the subtrahend	15.9
c. Misplacing decimal point in remainder	1.7
Number of errors analyzed–465	
3. Multiplication of decimals	
a. Misplacing decimal point in product	34.8
b. Computational errors	25.6
c. Annexing and prefixing zeros	8.4
d. Omission of decimal point in product	6.6
Number of errors analyzed–1,814	
4. Division of decimals	
a. Misplacing decimal point in quotient	38.3
b. Computational errors	16.4
c. Handling of zeros in dividend or quotient	14.7
d. Omission of decimal point in quotient	9.5
Number of errors analyzed–3,751	

On the basis of the data secured through a systematic case study a statement can be made of what the nature and underlying causes of the deficiency seem to be and the kinds of remedial measures that should be undertaken. Treatment cannot be effective unless it is guided by the results of a diagnosis.

While the busy classroom teacher often does not have the training to administer an individual diagnostic test nor the time

BRUECKNER DIAGNOSTIC TEST IN FRACTIONS

INDIVIDUAL DIAGNOSTIC RECORD SHEET—ADDITION

Name________________________________ Grade____________ Room__________

School______________________________ Date__________________________

Diagnosis	Summary	Diagnosis	Summary
I. Lack of Comprehension of Process		IV. Computation Errors	
a. Adds numerators and denominators		a. Addition	
b. Adds numerators, multiplies denominators		b. Subtraction	
c. Numerator added without changing to common denominator. Either denominator used in sum		c. Division	
II. Reduction to Lowest Terms		V. Omitted	
a. Fraction not reduced		VI. Wrong Operation	
b. Denominator divided by numerator		VII. Partial Operation	
c. Denominator and numerator divided by different numbers		a. In adding mixed numbers adds only fractions	
III. Difficulties with Improper Fractions			
a. Not changed to mixed numbers		VIII. Changing to Common Denominator	
b. Changed but not added to whole number		IX. Other Difficulties	

First indicate by number opposite each row the types of errors made on each example that was missed. For example Ia means that the pupil adds numerators and denominators.

Then summarize under "Summary" the total number of times each difficulty was found.

	Examples				
Row	1	2	3	4	5
I					
II					
III					
IV					
V					
VI					
VII					
VIII					

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EDUCATIONAL PUBLISHERS, Inc.
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to do so in the course of regular instruction, the nature of these tests and the diagnostic procedure used in applying them should be known by all teachers. The methods of diagnosis should be adapted and applied in the study of the work of any pupil whose work is seriously deficient. Unless his faults are known, remedial instruction cannot be effectively planned.

One page of the record blank of the Brueckner Diagnostic Test in Fractions is given on page 486. A manual that describes in detail the methods of diagnosis is provided with the test.

The names of available clinical diagnostic tests in arithmetic and their publishers are as follows:

1. Brueckner Diagnostic Test in Whole Numbers (Educational Test Bureau)
2. Brueckner Diagnostic Test in Fractions (Educational Test Bureau)
3. Brueckner Diagnostic Test in Decimals (Educational Test Bureau)
4. Buswell-John Diagnostic Tests for the Fundamental Processes in Arithmetic (Public School Publishing Co.)

Errors in per cent found by an analysis of 11,735 mistakes in diagnostic test papers of 405 seventh grade pupils in a midwestern metropolitan area are given below. The list does not include strictly computational errors as such. They were the single most frequent cause of error in all processes. Many of these errors indicate lack of understanding of the items tested.

1. In changing decimals to per cents:
   (a) Drops the decimal point and annexes the % symbol
   (b) Copies number and annexes % symbol
   (c) Changes decimal to equivalent fraction and annexes % symbol
   (d) Omits integers in mixed decimal and changes decimal to per cent
2. In changing per cents to decimals:
   (a) Merely drops % symbol in answer without changing to hundredths
   (b) Moves the decimal point to the left incorrectly
   (c) Inserts unnecessary zeros
   (d) Drops % symbol and annexes zeros

3. Changing common fractions to per cent:
   (a) Lacks knowledge of per cent equivalents
   (b) Adds % symbol to fraction without changing its form
   (c) Divides numerator of fraction by denominator but fails to carry work to hundredths
4. Changing fractions to hundredths and to per cents:
   (a) Merely copies numerator and writes as a two-place decimal
   (b) Copies entire fraction and writes as hundredths
   (c) Multiplies numerator by denominator
   (d) Errors in changing fraction to hundredths
5. In finding a per cent of a number:
   (a) Adds % symbol to answer
   (b) Divides the number
   (c) Makes errors in changing per cents to decimals
6. In finding what per cent one number is of another:
   (a) Divides base by percentage
   (b) Fails to express quotient as per cent
   (c) Multiplies base and percentage
   (d) Makes errors due to faulty manipulation of decimals
7. In finding a number with a per cent given:
   (a) Multiplies numbers representing rate and percentage
   (b) Divides rate by percentage
   (c) Divides percentage by rate, disregarding decimal form
   (d) Makes errors in manipulation of division of decimals
   (e) Subtracts numbers representing rate and percentage
   (f) Unable to manipulate fractions after changing rate to equivalent fraction.

2. *Informal case-study procedures*

Less formal procedures may also be applied by the teacher whenever a pupil does not do well on an analytical diagnostic test administered to a class. The following methods for determining difficulties in computations and problem solving in the course of daily work are helpful:

1. Examine the student's written work to determine faults, errors, incorrect procedures, poor form, etc.
2. Have the pupil work the incorrect examples again on another paper to see if the fault persists. Observe also

his methods of work, his attitudes, and symptomatic behavior.

3. In case of doubt have the pupil do the work aloud and observe his thought processes. Record illustrations of his procedures.
4. In case of doubt ask the pupil questions to get at subtle hidden thought processes and difficulties that the pupil may not be able to express orally. Test especially his understanding of the principles underlying arithmetic and algebra discussed in Chapters 5 and 12.
5. If you identify some apparent weakness in an under-lying process, for example in the subtraction used in division, it may be necessary to administer a diagnostic test in subtraction to see how serious the problem is. The disability may be specific or general in nature and cases should be dealt with according to the needs revealed by diagnosis.
6. Repeat the above steps for any or all operations where there is difficulty.

An informal procedure for diagnosing reading difficulties and other faults a pupil may have in solving verbal problems is:

1. Analyze his written work to determine if correct methods are used in solving problems and the extent to which computational errors are made.

2. Diagnose control of special types of reading skills involved in problem solving as follows: Take a set of four or five problems, preferably part of a test, in a textbook for the student's grade level. Ask questions similar to the following about each of the problems and note the pupil's responses:

a. Read the problem aloud for me (Observe quality of reading and look for possible evidence of reading difficulty)
b. Are there any words in the problem that we do not understand? (Test vocabulary and use of dictionary)
c. What does the question in the problem ask us to find? (Locating question)
d. What facts are given in the problem? (Analysis of problem, note approach)

e. How will you find the answer to the problem? Why do you think that is the way? (Dependencies among facts given and method of solution to use)

f. Make an estimate of what the answer will be. (Method of estimation)

g. Now find the answer. (Notice efficiency of methods of work)

d. Factors Associated with Lack of Success in Mathematics

The Sources of Difficulties in Learning Arithmetic

The most common sources of difficulty in number operations are the following:

1. Lack of understanding of the number system and of the ways in which it operates in computational procedures
2. Lack of knowledge of the basic number facts leading to guessing and random incorrect responses
3. Lack of understanding of the meaning of number operations and of the various steps involved in solutions
4. Inability to perform computations with reasonable speed and accuracy
5. The use of inefficient unsystematic procedures in making computations.

In addition to the above the following frequently are sources of difficulty in problem solving:

1. Inability to sense quantitative relations in verbal problems
2. Limited range of experience in dealing with arithmetic in social situations
3. Weakness in the special reading skills required in problem solving
4. Limited range of vocabulary of quantitative terms
5. Lack of knowledge of essential facts, rules, formulas, and information.

All pertinent data available in school records should be considered when diagnosing the work of a pupil, including his school history, his social background, his mental ability, his health and

physical condition, his interests, and his personality traits. When the pupil lacks interest in arithmetic and is unwilling to make the effort required to succeed, consideration should be given to the steps that can be taken to change his attitude, something that is often very difficult to bring about. Sometimes special adjustments of curriculum and instruction will be necessary to meet the situation, as shown on pages 498–499.

Wrenn[7] listed as follows the sources of the major difficulties of pupils in algebra:

> The major student difficulties in algebra may be traced to the following sources: (1) inadequate preparation, (2) inadequate comprehension of algebraic concepts, (3) failure to acquire speed and accuracy in the use of fundamental processes, (4) inadequate mastery of algebraic processes, (5) failure to acquire an effective technique for handling verbal problems, (6) failure to build up an attitude of confidence through a knowledge of how to check work, (7) inadequate understanding and mastery of short cuts, (8) failure to secure maximum profit from classroom instruction, (9) inefficient study techniques, (10) failure to develop a functional vocabulary and a proper appreciation for significant symbolism, (11) inadequate comprehension of the concept of functional dependence and the importance of its implications, and (12) lack of effective organization of information for reviews and tests.

To the above should be added the inability of the student to translate as a verbal statement an expression or equation stated in algebraic form. This point is discussed in detail in Chapter 12.

Characteristic Differences between Good and Poor Achievers in Problem Solving

A number of investigations have been made at the University of Minnesota to determine characteristic differences between good and poor achievers in problem solving. The areas in which highly significant differences were found and also areas in which there was no significant difference are given in Table XVI on page 492.

[7] *Encyclopedia of Educational Research* (W. S. Monroe, Editor), p. 721. New York: Macmillan, 1950. By permission of The Macmillan Company.

Table XVI
Differences Between Good and Poor Achievers in Problem Solving

A. *Differences Highly Significant*
  1. Psychological factors
    a. General reasoning ability
    b. Non-verbal mental ability
    c. Delayed and immediate memory
    d. General language ability
    e. General reading level
  2. Computation abilities
    a. Skill in fundamental operations
    b. Ability to estimate answers of examples
    c. Ability to see relations in number series
    d. Ability to think abstractly with numbers
  3. Problem solving reading skills
    a. Steps in problem analysis
    b. Finding the key-question of the problem
    c. Estimating answers to problems
    d. Ability to read graphs, charts, tables
    e. Range of information about arithmetic uses

B. *No Significant Differences*
  1. Range of general information
  2. Gates tests in general reading
    a. Grasp of central thought
    b. Prediction of outcomes
    c. Following directions
    d. Reading for details

The general conclusions to be derived from the above data may be stated as follows:

1. In such psychological factors as general mental ability, delayed and immediate memory, language level, and general reading ability, the differences between good and poor achievers in problem solving are highly significant in favor of the good achievers, that is, the scores for good achievers excel those of poor achievers.

2. The differences in skill and fundamental operations and ability to estimate answers also are *highly significant* in favor of the good achievers.

3. The differences in four basic reading skills peculiar to arithmetic as a special field in group A3 in the table and in range of

information about arithmetic also are highly significant in favor of the good achievers.

4. The differences between good and poor achievers in problem solving, in range of general information, and in the four general reading skills measured by the four Gates Silent Reading Tests are shown *not* to be significant. The conclusion is in sharp contrast to the third conclusion stated on page 492.

These results show that the ability to solve typical verbal problems, such as are found in standard tests of problem solving, depends largely on the general mental maturity of the pupil, his informational background, his ability to perform the necessary computations, and the possession of certain skills in reading peculiar to the solution of such problems. The fact that there are no significant differences between good and poor achievers in problem solving on four tests of ability to read the general types of literary material found in the Gates tests shows that there are specialized reading abilities in arithmetic that should be developed through systematic instruction. These abilities do not result from general training in reading, and they do not grow out of mere practice in computation. They must be developed through direct teaching. The significant differences also found in range of information about arithmetic and in ability to read graphs, charts, and tables suggest that units of experience which enrich and extend meanings and give practice in the interpretation of various forms of representation of quantitative data contribute significantly to the improvement of ability to solve problems. The fact that there are important differences between good and poor achievers in mental ability and general language development shows that the teacher must adapt instruction in problem solving to the needs and capacity of the learners.

e. The Treatment of Difficulties in Mathematics

Difficulty of Determining Causes of Learning Difficulty

Effective techniques have been devised for measuring ability in mathematics and for identifying specific deficiencies—their nature and scope. It is one thing to discover them, but their

treatment often presents a serious problem because the causes underlying them are complex and cannot easily be isolated. Today it has become customary to speak of "factors related to learning difficulties" rather than specific causes. If certain factors within the learner himself are not taken into consideration in planning the instructional program, difficulties may develop. For instance, it is necessary to adjust to such items as the learner's mental level, physiological defects and handicaps, his background of experience, his attitudes, and his emotional reactions. Similarly adjustments must be made of the curriculum, instructional procedures, materials of instruction, and environmental factors to assure the success of the student in mastering the content of mathematics. In most cases of complex disability a number of these factors apparently are operative and allowance should be made for them.[8]

Principles of Remedial Instruction

When the instructional program is adjusted to the needs and abilities of students the number who do not make satisfactory progress in mathematics is reduced to a minimum. However, under existing conditions there are found in almost every class a few pupils who are encountering unusual difficulties in learning mathematics. Diagnostic procedures are used by teachers to determine the nature of the problem and the causes of unfavorable progress. The teacher has the responsibility of planning the type of remedial work that will be most likely to lead to improvement.

Space will not permit the detailed discussion of corrective procedures that the teacher may apply. First we shall discuss a group of general principles that may be regarded as basic in the management of an improvement program; and then present a generalized list of remedial measures that are practical and readily applied to particular cases:

1. Treatment, based on a diagnosis, should be individualized.

[8] For a detailed discussion of this problem the reader is referred to the volume by L. J. Brueckner and Guy L. Bond, *The Diagnosis and Treatment of Learning Difficulties*, New York, Appleton-Century-Crofts, Inc., 1955 Chapters 3 and 5 deal with the general problem; chapters 8 and 9 deal with the diagnosis and treatment of learning difficulties in arithmetic.

2. Secure the interested cooperation of the learner so that he will be likely to attack his problems aggressively and willingly. Explain to him the nature of his difficulty and its significance. Describe also the steps to be taken to bring about an improvement.

3. Attack specific deficiencies of the learner directly. Begin reteaching at the point where there is likely to be success in the corrective work from the start so that the learner will find satisfaction in the progress he makes. Give special attention to the treatment of reading disabilities.

4. Take steps to correct any physical, emotional, and environmental factors that are likely to interfere with progress.

5. Proceed on a tentative basis in the correction of weaknesses, and do not hesitate to modify the steps taken when progress is slow and uncertain. Make extensive use of manipulative materials and visual aids to make the work meaningful to the learner, especially those of low mental ability.

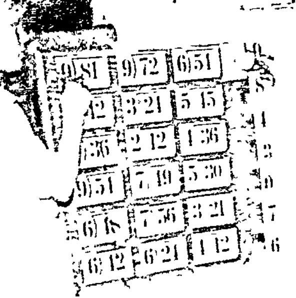
6. Select instructional procedures and materials that are of demonstrated value in making the operations meaningful for the learner. In general these procedures and materials should be based on the principles of teaching that have been discussed in preceding chapters.

7. Integrate the corrective and developmental program so that the learner will feel that he is not isolated and that he still is a member of the group.

8. Take steps to assure the growth of all aspects of the learner's personality. Do not focus on the correction of deficiencies to such an extent that positive values are neglected, such as interests, attitudes, and appreciations.

Types of Remedial Procedures

The types of remedial measures that have been used with success in remedial classes in mathematics and in educational clinics are listed below. The teacher can apply many of them quite readily in the classroom; some of the procedures will require the assistance of the services of specialists that are found in most school systems, including social workers, school psychologists, physicians, educational consultants, and guidance personnel.



John Mills School, Elmwood Park, Illinois

When remedial work reaches the stage where drill is helpful, a game device may serve the purpose.

Procedures Helpful in Corrective and Remedial Work

1. Medical care
   a. Correction of physical defects and limitations
   b. Glandular therapy
   c. Change in nutrition and diet
   d. Psychiatric attention for emotional disturbances and mental illnesses
   e. Planning of recreation, relaxation, and rest
2. Modification of curriculum
   a. Adapting contents of courses to the needs, interests, and level of ability of the learners
   b. Exploring vocational plans of the individual
   c. Use of rich social experiences to broaden the student's background of information and meanings
   d. Frequent application of mathematics to situations in community life
   e. Special remedial classes for students having unusual difficulty
   f. Full participation by slow learners in the activities of the class

3. Adapting methods of instruction
   a. Direct attack on specific areas of deficiency
   b. Self-diagnosis to make clear to the individual the nature and significance of his weaknesses
   c. Acceptance by learner of self-established goals to be achieved and responsibility for improvement
   d. Assurance of success in initial stages by beginning at a level where pupil will succeed
   e. Use of instructional procedures that will assure understanding of processes, including use of manipulative materials, visual aids, and clear explanations and demonstrations
   f. Close supervision of the pupil's methods of thinking and work to assure use of efficient procedures
   g. Careful gradation of sequence of steps to be mastered so that pupil will succeed and note progress made
   h. Distribution and variation of practice to avoid boredom and fatigue
   i. Informal tests to show the pupil the progress he has made
   j. Opportunity to apply what is being studied in a variety of situations as they arise in the classroom, school, and community
   k. Provision for maintenance of skills
   l. Use of recognition and approval to maintain purposes and drive
4. Changes in social environment as needed
   a. Correction of disturbing conditions
   b. Removal of learner from unwholesome home environment
   c. Necessity of an attractive classroom equipped as a learning laboratory
   d. Extensive use of community resources
   e. A classroom organization in which group dynamics function effectively
   f. Necessary therapy for associates whose behavior is not socially constructive
   g. Improvement of the attitudes of the individual toward mathematics and toward school and community life.

It has been shown repeatedly that it is possible to bring about great improvement in the ability of pupils to solve verbal problems in arithmetic. The developmental and remedial procedures that are used should, of course, be decided upon in the light of the needs of each individual as revealed by a critical analysis of his performance to determine his specific difficulties. In some cases it may be necessary only to improve the ability of the pupil to read problems; in others it may be necessary to try to improve his ability to perform the computations required; in others, experiences should be arranged to give meaning to the situations presented in problems. Similar activities adapted to the correction of other specific inadequacies should be provided. Sometimes a number of factors require attention. The important thing to remember is that a direct attack on identified sources of trouble begun at the level of the student's development yields large returns.

Specific procedures for improving problem solving of students are given below:

1. Discuss uses of number in social situations that arise, or in pictures and illustrations in textbooks to develop vocabulary and background experiences.

2. Give the student many opportunities to solve verbal problems that may arise in real social situations which in themselves present the need for computations. Have the pupil tell how he finds the answers and explain or give reasons for method of solution used.

3. Give the student abundant experience in reading and solving easy, meaningful, verbal problems in textbooks and workbooks, beginning at or slightly below his level of development. Keep the numbers small at the start. Gradually increase the difficulty of the vocabulary, computations, and situations.

4. Be sure to help the learner to understand the meaning of each number process by use of suitable materials and by exercises containing problems which embody the various types of situations or meanings each process implies. Textbooks often

provide suitable exercises which visualize the meanings of the various processes. Have pupil give original problems.

5. Have pupils use manipulative materials to demonstrate and work out solutions of simple problems so as to help them to visualize the situations and the relations involved. Have them tell orally about the procedures they use.

6. Plan reading experiences which will develop the special reading skills in problem solving listed in the outline on diagnosis. Take one skill at a time in the sequence listed. This is very important. See workbooks for special helps.

7. Emphasize vocabulary development by suitable exercises.

8. Do special work on measures and their use in social situations. Help pupils to develop and then use tables of measure. Demonstrate or visualize the processes of conversion from "large to small" and "small to large," a basic element of difficulty in problem solving.

9. Develop basic rules, formulas, and procedures through real situations, involving manipulation of representative materials, drawings, visualization and thinking through the relations involved; for example, perimeter, area, costs, interest, percentage, etc.

10. Use problems without numbers to help the pupil to learn to state in his own words how to find the answers.

11. Emphasize the need of accuracy in all computations. Teach pupils to go over their work to check it.

12. Teach pupils procedures for approximating and estimating answers to help them to see if their own answers to problems are sensible.

Questions, Problems, and Topics for Discussion

1. What procedures can be used to evaluate the various kinds of educational outcomes of instruction in mathematics that were presented on page 3?

2. Select some specific outcome. Then follow the steps described on pages 459 to 461 to develop some method of appraisal

3. What aspects of intellect should be considered in evaluating an outcome? How can one appraise the student's rate of work? his accuracy? the level of his ability? his methods of work?

4. Why is it important to consider the student's method of arriving at answers, and not only the answers found, in evaluating his performance?

5. Select from the list of possible appraisal procedures given on pages 464 to 466 those that you would apply in making a general evaluation of the effectiveness of the program in mathematics. Be ready to explain why you selected each procedure.

6. Look up the meaning of sociometric procedures and tell about their usefulness in the classroom.

7. Secure the results of some standardized test in arithmetic, algebra, or general mathematics. Analyze the data and prepare a report for class discussion.

8. What are the functions of aptitude and prognostic tests? Find out if and how they are used in local schools.

9. Prepare objective test items that exemplify different procedures for testing some specific ability in mathematics. The illustrative items on pages 467 to 468 may be helpful.

10. Explain and illustrate the various procedures for appraising the outcomes listed in the chart on pages 469–470.

11. What is meant by "levels of diagnosis"? What is the difference between the three levels? When are they applicable?

12. Discuss the steps to be followed in diagnosing a specific learning difficulty or disability in mathematics.

13. Secure the results of a general survey test and make a general diagnosis of the situation. Prepare a profile for some student and analyze the data.

14. What is meant by "simple retardation"? What is a "specific disability" case? What is a "complex disability" case? If possible, describe actual cases that you believe fall under each of the three categories.

15. What is meant by "analytical diagnosis"? Describe possible procedures to apply. Find out the kinds of analytical diagnostic tests that are included in textbooks and workbooks and how they are used in regular instruction. Some classroom teacher should administer one of the diagnostic tests published by the California Test Bureau and discuss the findings. What is meant by "component" skills?

16. What is meant by "case study" procedures? What standard diagnostic tests are available? What types of informal diagnostic procedures can classroom teachers apply in the course of regular instruction? Discuss with several teachers of mathematics the use of diagnostic procedures

17. How does an awareness by teachers of the types and causes of errors made most frequently by students assist them in the diagnosis of pupil difficulties? Can you illustrate the sources of pupil difficulties given on pages 487–488?

18. What are the characteristic differences between good and poor achievers in problem solving?

19. Why is it difficult to determine the specific causes of disability in mathematics? Why may reading disability be a major source of difficulty in mathematics?

20. What are the principles of remedial instruction that are discussed in this chapter? Would you add others?

21. Discuss the methods of applying the four major types of corrective and remedial procedures given on pages 496–497.

22. What specific procedures can be used to improve problem solving ability?

Suggested Readings

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Chapter 14

Mathematics for the Gifted Student

This chapter deals with means of enriching the curriculum for the superior student in mathematics. The major topics discussed include the following:

a. The need of enrichment
b. Our number system
c. Books for the mathematics library
d. Mathematical games
e. An introduction to conic sections
f. Geometric constructions
g. Miscellaneous means of enrichment.

a. The Need of Enrichment

Providing for Individual Differences

Preceding chapters give much emphasis to differences in ability and achievement of groups of students in the upper grades. To deal effectively with these differences, special attention must be given to the gifted student as well as to the slow learner. Gifted students have too frequently been the most neglected of any group. Much material designed particularly for the slow learner is now available. There is a limited amount of material specifically designed for the gifted student. The teacher of gifted students must, therefore, select material from a wide range of sources. The gifted student must be stimulated by

the material so that it will provide an incentive and a challenge within the range of his ability and background.

Fehr emphasized the need for the teacher to give special training to mathematically gifted students. He also gave the following characteristics of these students.

"In the United States there is now a tremendous shortage of mathematicians . . . Beginning at the elementary level, teachers must identify, motivate, and educate the largest number of children who are superior or gifted in mathematics.

"The selection must start early. An I.Q. above 120 as measured by standard tests is only one indication of possible mathematical ability. Other indications include extraordinary memory, ability to generalize, intellectual curiosity, persistent goal directed behavior, keen quantitative insight, facility in reasoning, and an unusually advanced knowledge of mathematics.

"The elementary school teacher who *knows* arithmetic, algebra, and geometry and is familiar with the literature in these fields available for use by children will readily find projects in mathematics to develop the gifted child."[1]

In recent years different means have been reported to determine the characteristics which enable teachers to identify gifted students. Several publications (15) (32) (50) and (54)[2] have dealt with general aspects of this problem.

McWilliams and Brown define superior students as ". . . the rapid learners in academic subjects. They are the pupils in the upper 20 per cent in general intelligence whose abilities lend themselves readily to intellectual pursuits" (32 p. 3).

The Objectives of Enrichment

Four major objectives of enrichment are:

1. To encourage students to work independently and learn directly from appropriate books and periodicals

2. To enable students to develop broader skills and to acquire more technical knowledge than the average student can assimilate

[1] Fehr, Howard F. "The Student Gifted in Mathematics," *NEA Journal* 43.222.

[2] These and similar numbers refer to titles in the bibliography at the close of this chapter.

3. To give the student the opportunity to explore, discover, and develop his interests and potentialities in mathematics

4. To challenge the student to work at the level of operation at which maximum growth in mathematics is possible.[3]

The second and third objectives are self explanatory. According to the first objective, the student should be encouraged to work independently. A prime function of school instruction should be to enable the student to identify and solve problems by himself.

The fourth objective states that the student should be challenged to work at a level at which he will have maximum growth in mathematics. If the same kind of challenge is offered to gifted students as is given to students of average or below average ability, the work is geared to such a low level of abstraction that the superior student is not challenged to have maximum growth in mathematics.

Means of Enrichment

A survey of the literature dealing with the problem of meeting the needs of the superior student shows that four methods or procedures are used. They are:

1. Acceleration
2. Segregation
3. Enrichment
4. A combination of any of the first three.

Acceleration means that the student will cover the work in less than normal time. McWilliams and Brown found that many junior high school teachers believe that mathematics is the one subject area in which acceleration is the best possible means of making suitable adjustment for superior students (32 p. 28).

Segregation implies that special classes are formed for the superior students. In such places as New York City and Long Beach, California, certain schools offer special opportunities for students of this type.

[3] Grossnickle, Foster E. "Arithmetic for Those Who Excel," *The Arithmetic Teacher*. 3:41.

Both photos Public Schools, Chicago, Illinois.

Work beyond the minimum requirements often provides enrichment.

The third plan calls for enrichment of the curriculum. Such activities as solving additional exercises and problems, individual and group reports on topics investigated, and participation in mathematics clubs represent a phase of enrichment of the curriculum in mathematics.

Acceleration and segregation are predominantly administrative procedures for dealing with a vital problem in learning. The activities listed under enrichment touch upon the problem of learning. These activities are favorable but inadequate. Therefore, there must be at least one more plan for dealing with the superior student in mathematics.

A fifth plan calls for differentiation both of the curriculum and of the level of abstraction at which the student operates. At many places in this text these two principles have been implemented. For example, the discussion of the work dealing with fractions showed how to differentiate the curriculum from the standpoint of subject matter. The slow learner should learn to add or subtract only fractions having social significance. Furthermore, he should be permitted to use exploratory and/or visual aids in arriving at his answers. The superior student should not necessarily be restricted to addition or subtraction of fractions having social significance. This type of student should not use supplementary aids, except in initial instruction, in dealing with these fractions.

Grouping within the class (sub-grouping) provides many or most of the advantages of segregation in the field of arithmetic. The groups formed within the class should be fluid and not static. There are many times in an arithmetic period in which the class should function as one group. Sub-grouping at such times would be unprofitable and undesirable. In a unit dealing with taxation, the class should function as a unit. It may be advisable to form sub-groups when dealing with such work as finding the tax rate or some other phase of taxation involving computation or problem solving.

Although we shall be concerned primarily in this chapter with enrichment for the superior student, it should be understood that enrichment is not intended solely for this type of student. Ideally, the teacher should provide enrichment for every student. Often

the fast learner needs new channels for his energy because he is not challenged by the level of the work when it is geared to fit the needs of the slower learner. In such cases the fast learners will profit most by the application of some of the means of enrichment described above.

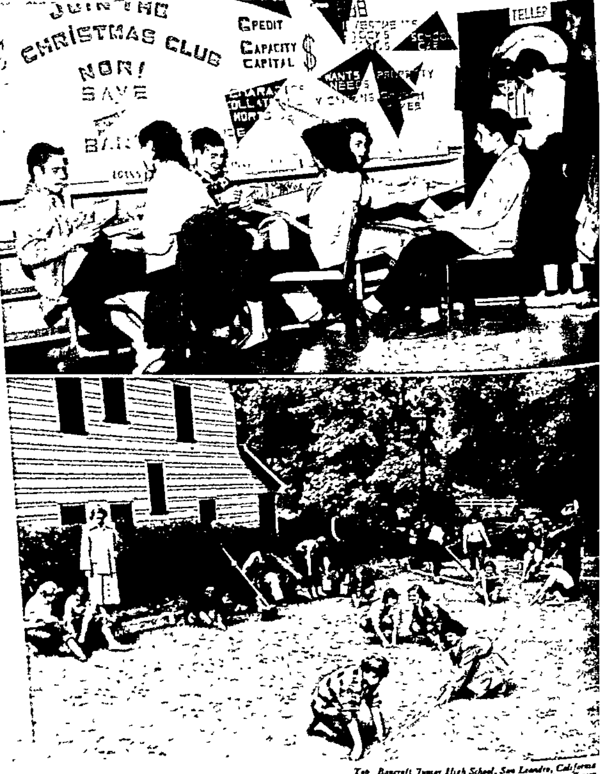
The library is the heart of an enrichment program. The bibliography at the end of this chapter has been selected with care. The titles listed approximate a minimum requirement for carrying on an enrichment program that is broad enough to challenge the great variety of students in today's schools. The acquisition of a library, no matter how large, does not assure an enrichment program. A set of units designed for self study probably is the most flexible means of providing enrichment for students. These units should be revised periodically on the basis of the results achieved by students working on them.

Types of Enrichment Experiences

Four methods of enriching the mathematics curriculum are:

1. At an appropriate time during a class discussion, a particular project may be assigned to a student whose work and background indicate that he is ready for it. For example, when the construction of the perpendicular bisector of a line segment is being introduced, the teacher may assign one of the students who has shown interest and ability in geometry the task of determining how this construction may be used to find the unknown center of a circle, or how to draw a circle through any three points not in a straight line. The student should be given a reference indicating where the information can be found and should make a brief report to the class. Under some circumstances, it may be better to ask for volunteers rather than to choose the student directly. Opportunities for assistance should be provided.

2. A challenge may be presented to the entire class. The teacher may have the assurance that only students above average in ability and interest would accept this challenge. Subsequent check-ups and hints to individuals may supply the additional impetus necessary to obtain desirable results. For example, after discussing a problem requiring both multiplication and division,



Top Bancroft Junior High School, San Leandro, California
Bottom Bloomfield Junior High School, Bloomfield, New Jersey

Socializing a mathematical situation is a form of enrichment for the entire class, not only the gifted.

the teacher may remark that the use of the slide rule might reduce the time required for computation to one-tenth of that required by the usual pencil and paper method. If slide rules and instruction booklets are then made available to interested students, a favorable learning situation has been established. By keeping in touch with the progress made, the teacher may also be able to provide some necessary aid. Students who acquire some proficiency with the slide rule should be permitted to use it in appropriate situations in homework and quizzes. The choice of such situations should be made so that the student becomes aware of the limitations of the slide rule as well as its advantages.

3. A topic may be introduced to the entire class in a simple form. The teacher should then inform the students where more advanced discussions of this topic may be found. The simplest forms of magic squares may profitably be introduced to an entire class. Information may then be given where more advanced work is available such as reference 17 in the bibliography at the end of this chapter. Students who pursue the topic further should be encouraged to make a brief report to the entire class of what they have learned.

4. There are many activities outside of the classroom that provide enrichment. These include:

a. The preparation of mathematical exhibits
b. The creation and/or performance of mathematical plays
c. The participation in mathematical contests
d. The maintenance of a "math" bulletin board or table
e. Helping in the laying out of tennis courts or other athletic fields
f. Making a map of the school grounds.

The student frequently should be impressed with the fact that *the highest type of learning results from working alone and not from consulting any other person.* While the learning of the facts and ideas discussed and emphasized in class should not be minimized, all students should be encouraged to strive for the higher goal of learning which results from independent study. Consequently, students should be encouraged to search for vital topics not discussed in class and to investigate them. A report, competently

handled, on a topic discovered by the student without teacher assistance should be valued above a report on any of the assigned units. Independent reports by students require no preparation on the part of the teacher, but time must be taken to evaluate them properly.

Some of the above procedures are related and may be used to supplement each other. The remainder of this chapter gives a broad cross section of a variety of topics and materials for enriching the experiences of students who show marked ability in dealing with mathematics.

b. Our Number System

The number system offers one of the most natural fields of enriching the standard curriculum. Our number system is the culmination of the efforts of many people over thousands of years. Some suitable topics for investigation are enumerated below.

1. A history of the highlights of our number system can be a most profitable topic. Such a topic should include a discovery of zero and the place notation. Different number systems which have existed in other civilizations have their distinguishing characteristics. The study of several of these might provide interesting contrasts.

2. Special properties of our number system are very enlightening, such as those involved in casting out nines and elevens, and in rules for divisibility.

3. The structure of our decimal number system is of great interest. This probably is best studied by investigating number systems which have a base of eight, twelve, or any other given value.

Some suggested references with comments on phases in which these number systems excel are discussed below:

1. *Number Stories of Long Ago* (44) contains excellent material on the history of our number system and its development. The language is suitable for the junior high school level. Much of the material in this book may be used to integrate mathematics with the social studies.

2. *Numbers and Numerals* (46) begins with counting and proceeds with a discussion of names and symbols for quantities. Unusual properties of numbers, historical facts, and the relationship of numerals to computation also are discussed in an interesting and elementary manner.

3. *The Wonderful Wonders of One-Two-Three* (45) is very readable and gives many interesting facts about numbers. The origin of "three cheers," mathematical properties of numbers, and magic squares are several of the topics included.

4. *History of Mathematics* (40) can be used at the upper grade level. By studying this book a student can obtain much interesting information to improve his mathematical background.

5. *New Numbers* (4) presents the case for a number system with twelve symbols instead of ten symbols. Much of this book is too technical except for the most gifted junior high school student. However, every time this text makes some student aware that the ten digits in our number system are arbitrary and not necessarily the most ideal number, it will justify its inclusion in a school's library. The Duodecimal Society of America has published pamphlets and bulletins on the advantages of a *duodecimal* (base 12) number system.

6. Standard encyclopedias such as *The World Book, Compton's Pictured Encyclopedia*, or *Britannica Junior*.

7. An excellent bibliography on different phases of number is given in *The Mathematics Teacher* of December 1953.

c. Books for the Mathematics Library

Books Written for Students at the Junior High School Level

As particular topics are discussed later, specific references are made to appropriate books. The following books do not fall into any one area of subject matter but are written on a level suitable for use in the junior high school.

1. *How Much and How Many* (9) is a story of weights and measures written in a very readable style. It will supply enrichment for the student of average ability as well as for the more gifted student.

2. *Flatland* (1) is an excellent introduction to the nature of the fourth dimension. It is not necessary to read the entire book to obtain the main ideas that are presented.

3. *Fun with Figures* (14) provides a variety of activities for students at all levels of ability. Its sub-title, "Easy Experiments for Young People," suggests its approach.

4. *Patterns of Polyhedrons* (16) provides patterns for the construction of the various polyhedrons. Such constructions afford an excellent activity dealing with the mathematical phase if the work is understood.

5. *Winter Nights Entertainments* (2) is essentially a book on mathematical recreations. It is written on a level that is suited to a wide range of junior high school students.

6. *Diversions and Pastimes* (3) is an extension of the *Winter Nights Entertainments.*

7. Selected *Yearbooks* of the National Council of Mathematics Teachers (33, 34, 35) have many topics of direct interest to students even though the bulk of the materials are of interest only to teachers.

8. One or more sets of standard encyclopedias may be consulted.

9. *The World Almanac* should be available for reference because of its numerical data.

An excellent list of reference books on different phases of mathematics particularly suitable to high schools appears in *The Mathematics Teacher* of February, 1954. Most of these books have material that is suitable for the junior high school.

d. Mathematical Games

Mixing Fun and Mathematics

Mathematical games offer an opportunity to mix fun with mathematics. A few samples of such games are given below.

A simple game with numbers may be played between two contestants in the following manner. The first contestant chooses any number from one through seven. The second player also chooses a number from one through seven and adds his choice

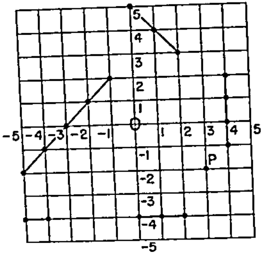
to the number selected by the first player. Then each player adds certain selected numbers in turn. The player obtaining the sum of 50 wins the game. An alert student will soon learn that the player arriving at the sum of 42 first can always win since the next player must then arrive at a sum ranging from 43 through 49. It is possible to obtain a sum of 50 by adding 7 or less to any number in this range.

The next step in the successful analysis is to realize that the first player to obtain a sum of 34 can be certain to reach a sum of 42. By continuing in this manner, the student can determine that the key numbers are 2, 10, 18, 26, 34, and 42. By starting with 2 and obtaining each of the key numbers in succession, the first player is certain to win.

A teacher may introduce this game by playing against the entire class. The gifted student should then be challenged to invent similar games with different limits and final sums with the ultimate challenge of formulating a rule by which all such games can be won. This generalization is most readily stated in the language of algebra but can be stated clearly in words without the use of algebraic symbols. A discussion of a similar game with variations, called the "Battle of the Numbers," can be found in *Mathematical Recreations* by Kraitchik (26).

Nim is one of the most venerable of all games for two players. It may be played with many variations, but there always should be two or more piles of counters. Players may remove counters from one or more of these piles. The player wins when he picks up the last counter. A simple form of this game involves piles of seven, five, and three counters. Each player may pick up one, all, or as many as he may prefer to choose from any one pile. A player should soon learn that to leave his opponent with two piles of one counter each will guarantee victory. To leave an opponent with two piles of two counters each will also assure a win. With such a small number of counters, it is not difficult to learn all the key positions. A complete mathematical analysis of this game involves the binary number system and is discussed rather fully in *Elementary Number Theory* (52).

A variation of this type of game may be played with a single pile of 15 counters. Each player may remove one, two, or three



counters at a time. In this case, the winner is the player who finishes with an even number of counters.

A game called *Naval Combat* is excellent for students who are familiar with or who are learning the system of rectangular coordinates usually introduced in a first course in algebra. In this game, each of two players locates five naval ships—a battleship, a cruiser, two destroyers, and a submarine—on his own graph, ten units square. The *origin*, O, is in the center of the graph and each axis extends from -5 to $+5$.

Ships are represented by straight lines passing through two, three, four, or five consecutive points with whole number (positive or negative) *coordinates*. The coordinates of a point are the x- and y-values of that point. Thus, the coordinates of the point P are $+3$ and -2, usually written as $(3, -2)$. The x-value is always written first. Ships may be placed vertically, horizontally, or diagonally. The battleship is represented by a line containing five consecutive points with whole number coordinates. The cruiser is represented by four consecutive points; each of two destroyers is represented by three consecutive points; and the submarine, by two consecutive points.

The first player fires a salvo of five shots by giving the coordinates of points with whole numbers, such as $(-5, 4)$, $(1, 4)$, $(4, -1)$, $(5, 2)$ and $(-3, -4)$. The second player locates or *plots* these points on his own graph which contains his fleet. If the second player had ships placed as those shown above, he would

then report that the cruiser and one destroyer had each been hit once without indicating which shots were hits. The second player then fires his five salvos. A ship is sunk when all of its points have been hit. The first player to sink completely the other player's fleet wins. While victory depends largely on chance, there is an opportunity for analyzing situations and estimating where the ships are located.

Many other interesting and suitable games are described in the various books on mathematical recreations listed in the bibliography at the end of the chapter.

Puzzles

The variety of material in mathematical recreations is almost unlimited. There are many simple puzzles which require careful analysis but use only simple arithmetic skills. Other recreations may involve algebraic or geometric skills of a wide range of difficulty. Every teacher should collect a set of puzzles which will appeal to students of various levels of ability and interest.

The following puzzle requires only simple arithmetic. However, in order to deal with it effectively, one must use sound analysis.

Mr. X went into Mr. Jones' store with a \$10-bill to make a \$2-purchase. Since Mr. Jones had no change, he went next door and obtained the change from Mr. Smith. After Mr. X left with merchandise worth two dollars and eight dollars in change, Mr. Smith discovered that the \$10-bill was counterfeit. When so informed, Mr. Jones immediately gave Mr. Smith a genuine \$10-bill for the counterfeit bill. How much did Mr. Jones lose on the entire transaction?

A number of conflicting answers usually will follow. An easy way to solve this puzzle is to assume that Mr. Jones had \$50 (or any other amount) in his cash register just before Mr. X entered. By analyzing each step of the transaction it then can be seen that Mr. Jones should have \$42 in his register (plus a worthless \$10-bill). Consequently, he lost \$8 in cash and merchandise worth two dollars.

Does 4 = 5?

A puzzle may frequently highlight a very important mathematical fact. Consider the following:

An identity:	$16 - 36 = 25 - 45$
Add $\frac{81}{4}$ to each side:	$16 - 36 + \frac{81}{4} = 25 - 45 + \frac{81}{4}$
Factor:	$(4 - \frac{9}{2})^2 = (5 - \frac{9}{2})^2$
Take square root of both sides:	$4 - \frac{9}{2} = 5 - \frac{9}{2}$
Add $\frac{9}{2}$ to each side:	$4 = 5$

Since the first statement is true and each of the following statements is derived by apparently valid algebraic manipulations, the paradox is difficult for a beginning student in algebra to explain.

This puzzle may be used to highlight the fact that even though the squares of two numbers are equal, the numbers may be different. It is true that they must have the same absolute value (value without signs) but their signs may be opposite. Thus, 2 and -2 are quite different but their squares are the same number. If one examines the third step in the puzzle more carefully, he should see that $(-\frac{1}{2})^2 = (\frac{1}{2})^2$. Even though $(-\frac{1}{2})^2 = (\frac{1}{2})^2$, it is not possible to conclude that $-\frac{1}{2} = \frac{1}{2}$. Therefore, if $A^2 = B^2$, either $A = B$ or $A = -B$. Further information is required to determine which of these statements is true although it is possible under some circumstances for both statements to be true. This fact is frequently written as $A = \pm B$. To those junior high school students whose curriculum includes quadratic equations, this puzzle is very useful.

Many Answers Possible

The following problem requires but a few algebraic principles although it touches on a branch of mathematics not normally covered in elementary algebra. A man cashed a check and received dollars for cents and cents for dollars. As a result of this mistake, he received \$19.80 more than he should have received. What was the amount of the check?

If x is the correct number of dollars and y is the correct number of cents, it follows that:

$$(100y + x) - (100x + y) = 1980$$
$$\text{or } 99y - 99x = 1980$$

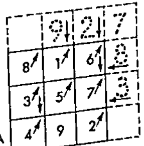
Divide by 99:
$$y - x = 20$$

This is a single equation in two unknowns, but it has the additional property that the solution must be in positive whole numbers. It is then fairly evident that if 21 is substituted for y and 1 for x, the equation is true. A check of \$1.21 would satisfy the conditions of the problem. There are many other solutions to this problem such as $y = 22$ and $x = 2$; $y = 23$ and $x = 3$, and the like. Such equations are called *diophantine* equations and it is characteristic of them to have an unlimited number of solutions. A systematic treatment of such equations is beyond the level of the junior high school student. There is, however, a readable historical and mathematical discussion of such problems in *Number Theory and Its History* (38).

Magic Squares

A *magic square* is an arrangement of numbers in a square so that the sum of each row, column, and diagonal is the same. In a classic magic square the numbers are consecutive beginning with 1. Magic squares date from before 2000 B.C. Each number in a magic square occupies a *cell*. Magic squares contain either an odd or even number of cells on a side. There are many different methods of making magic squares. Heath has described some of the easiest ways to make magic squares in *Mathemagic* (17).

The method illustrated in square A, with three cells on a side, may be used to construct any magic square with an odd number of cells. To make such a magic square, always begin by placing the number 1 in the middle cell of the upper row. The normal direction of procedure is diagonally upward, to the right. When the next cell



A

to be filled lies outside the square and above it as does the cell occupied by the dotted figure 2 in square A, proceed to the cell at the bottom of that column and write 2. In the next cell diagonally upward and to the right, you should write 3. Since this cell is outside of the square and to the right, place 3 in the first cell in that row as shown. The next cell diagonally upward and to the right is occupied by 1. In such a situation proceed to the cell immediately below the one containing the last number written, 3, and write the number 4. The numbers 5 and 6 will now complete the cells along the diagonal by proceeding upward and to the right. The cell for the next number, 7, lies outside the square in both a row and a column. In this case proceed to the second cell from the top in the right hand column and write the number 7 as shown. Since the cell for 8 lies outside the square and to the right, proceed to the first square in this row and write the number 8. The remaining square may now be filled with the number 9 by the process of elimination or by following the indicated pattern. The arrows indicate the pattern to follow. All magic squares with an odd number of cells on a side may be constructed by following this pattern. The reader may check his understanding of this method by completing square B.

B

17		1		
23				16
4	6			
10			21	
11	18		2	9

If the numbers in a magic square having an odd number of cells on a side are consecutive and begin at 1, the numbers are always consecutive along the diagonal from lower left to upper right. The constant sum of a magic square which contains consecutive numbers beginning at 1 may be found by the formula, $S = \frac{1}{2}n\,(n^2 + 1)$, in which n is the number of cells on a side. In square B, n is 5 and S is 65.

Though this formula holds for the sum of even-celled as well as odd-celled magic squares, the construction of an even-celled magic square is very different from an odd one.

To construct a magic square with four cells on a side, first draw the diagonals as shown in square C. Starting at the cell at the upper left, count consecutively and write numbers in the cells not cut by a diagonal as illustrated in square C. The upper left cell is vacant because it is cut by a diagonal. Now place the number 16 in this same cell at the upper left and count backwards to fill in the vacant cells. Square D is the magic square obtained by this method.

C

	2	3	
5			8
9			12
	14	15	

D

16	2	3	13
5	11	10	8
9	7	6	12
4	14	15	1

A magic square with eight cells on a side may be constructed by a method similar to that given for a magic square with four cells on a side. The interested reader can fill in the empty cells in square E.

E

64	2	3	61	60	6	7	57

The method described for a square of four cells on a side does not apply to a magic square having six cells on a side. Further information on this type of square is available in Heath (17).

A magic square usually affords the student enjoyment in dealing with numbers. After he has learned to make the magic squares already described, he should explore some of the different groupings in which the constant sum may be found. In square D, for example, there are at least a dozen different groups of four numbers having a sum of 34. A few such groups, easily found, are 16, 4, 1, 13; 2, 3, 14, 15; 16, 5, 3, 10; and 11, 7, 6, 10.

Magic Squares with Fractions

Magic squares also afford excellent practice in addition and subtraction of whole numbers, common fractions, and decimals. If certain cells are filled, the student may complete the square by addition and subtraction. Trial and error should soon determine which cells can be left vacant so that it will be possible for the student to complete the magic square by addition and subtraction.

Square F shows how it is possible to use common fractions to make a magic square. Any whole number greater than 1 may be used as the denominator of the fractions. The numerators are the numbers in a magic square, as in D. Each fraction may then be expressed in its simplest form. The numbers in square F are obtained by using 12 as a denominator and the numbers in square D as numerators. The first number obtained in this manner is $\frac{16}{12}$ which may be expressed as $1\frac{1}{3}$. It is possible to make a magic square with decimal fractions in a similar manner.

A magic square containing negative numbers may be constructed by subtracting the same number from the number in each cell of a magic square.

F

$1\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{4}$	$1\frac{1}{12}$
$\frac{5}{12}$	$\frac{11}{12}$	$\frac{5}{6}$	$\frac{2}{3}$
$\frac{3}{4}$	$\frac{7}{12}$	$\frac{1}{2}$	1
$\frac{1}{3}$	$1\frac{1}{6}$	$1\frac{1}{4}$	$\frac{1}{12}$

Algebraic Magic Squares

The beginning algebra student should be encouraged to derive rules for making magic squares using literal numbers, such as a, m, or r. In square G the top row contains any three numbers a, b, and c. Place x in the center cell. Now determine the quantities which must be in the two lower corner cells to produce the sum $a + b + c$.

G

a	b	c
	x	
$a+b-x$	y	$b+c-x$

Next, place y in the center cell of the lower row and remember that the sum of the lower row must equal the sum of the middle column. The equation stating this fact is:

$$(a + b - x) + y + (b + c - x) = b + x + y$$

Solving for x: $$x = \frac{a + b + c}{3}$$

These algebraic results may be interpreted verbally to mean that a magic square formed by using literal numbers with three cells on a side may be constructed by performing the following steps:

1. Place any three numbers in the cells of the top row.
2. Place one-third the sum of these numbers in the center cell of the second row.
3. Complete the square by addition and subtraction.

This type of square is excellent for negative and literal numbers, and for fractions.

Square H may be used to present a worthwhile challenge to a gifted student in the first year of algebra. It must be understood that a, b, c, and d are any four numbers. The sum of this square is $s = a + b + c + d$. The numbers e, f, and g must be chosen so that $a + e + f + g = s$. The numbers h and i must be chosen so that $d + h + i + g = s$. The proper choice of x and y will now make it possible to complete the square by addition and subtraction. A correct analysis can lead to the following results for x and y.

a	e	f	g
b		i	x
c	h		
d			y

H

$$x = d + h - f$$
$$y = e + f - d$$

Other correct results are possible.

Certain other types of magic squares with restrictions on the numbers used are discussed by Rich.[4]

[4] Rich, Barnett "Additive and Multiplicative Magic Squares," *The Mathematics Teacher*, 44:557–559.

Solving Problems without Using Computation

Puzzles requiring reasoning without computation should not be overlooked. An illustration typical of this category is:

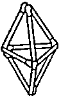
An employer wishes to hire one of three men and devises a test for this purpose. The three candidates for this job are placed at a table facing each other. The employer touches each man on the forehead with his finger and then tells them that each man may or may not have a black spot on his forehead. The three men are then instructed to fold their arms if they see a black spot. Each man immediately folds his arms since the employer has put a spot on each forehead. Thus, each man sees two black spots but he cannot see if there is a black spot on his own forehead. The employer then tells the candidates that the first man to prove that his own forehead does or does not contain a black spot should report to the office next door.

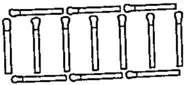
The three men sit around the table for about one minute when the fastest thinker arises and goes next door announcing that he has a black spot on his forehead. How does he know?

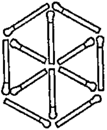
For convenience call the three men A, B, and C, with A being the winner. A's first reaction is one of indecision. The folded arms give no direct information since B's arms would be folded for C's black spot and vice versa regardless of the state of A's forehead. After some meditation, A arrives at the major conclusion that B and C are also undecided. He then realizes that B and C can be undecided only if he, A, has a black spot. With no black spot on A's forehead, B could look at C's folded arms and A's blank forehead and immediately know that he, B, must have a spot. In short, A then realizes that if he has no spot, both B and C would realize almost immediately that each of them has a black spot. Since they do not arrive at this decision, A concludes that he must have a black spot. Many puzzles of this type are available in most books on mathematical recreations.

Geometric Puzzles

Many puzzles calling for arrangements with match sticks have geometrical significance. A puzzle of this kind can be given to







determine how nine matches can be arranged so as to form the largest possible number of equilateral triangles. The answer is seven with three of the matches forming the base for a double tetrahedron as illustrated.

Another similar puzzle involves the six equal areas formed by the thirteen matches as shown. The question is how to obtain six equal areas with one less match. The third drawing shows how this may be done.

This discussion makes no attempt to cover all the suitable topics from the vast field of mathematical recreations. A representative sample is given to remind the teacher of the wealth of material that is available in the literature in this field.

Computational Shortcuts Based on Aliquot Parts

Computational shortcuts are possible by use of aliquot parts of a number. If one number is divisible by another number, the smaller number is an *aliquot part* of the larger number. Thus, 6 is an aliquot part of 18. The use of aliquot parts of 10, 100, or 1000 frequently is useful in multiplication and division Since 25 is an aliquot part of 100, one may multiply a number by 25 by first multiplying by 100 and then dividing this product by 4. Similarly, one may divide by 25 by first dividing the number by 100 and then multiplying the quotient by 4.

1. $25 \times 78 = \dfrac{100 \times 78}{4} = 1950$

2. $\dfrac{642}{25} = \dfrac{642}{100} \times 4 = 6.42 \times 4 = 25.68$

Similar shortcuts in multiplication and division are possible for the aliquot parts of 100 or 1000.

$$50 = \tfrac{1}{2} \text{ of } 100;\ 12\tfrac{1}{2} = \tfrac{1}{8} \text{ of } 100;$$
$$16\tfrac{2}{3} = \tfrac{1}{6} \text{ of } 100;\ 125 = \tfrac{1}{8} \text{ of } 1000.$$

An exercise in multiplication and division using aliquot parts should be a valuable means for enrichment of number work for the superior student. Such a shortcut, however, has little value if the student does not understand the process.

Computational Shortcuts Based on Algebraic Identities

Computational shortcuts based on algebraic relationships make interesting and instructional material. Any two-place number ending in 5 may be represented as $10x + 5$ in which x is the tens' digit. The square of any two-place number ending in 5 may be represented as follows:

$$\begin{aligned}(10x + 5)^2 &= 100x^2 + 100x + 25\\ &= 100x\,(x + 1) + 25\end{aligned}$$

Many students can perform the manipulations as indicated without being able to understand or interpret the results. The ability to interpret the results of algebraic manipulations is at least as important as the ability to perform these operations. For example, the above relationship may be interpreted as follows: Any two-place whole number ending in five may be squared by multiplying the tens' digit by one more than itself and annexing 25 to this product. To square 65, multiply 6 by 7 and annex 25 to this product. This answer, 4225, may be checked by ordinary multiplication.

The following relationship may be used to square any whole number which has a value near 50.

$$(50 + x)^2 = 100\,(25 + x) + x^2$$

To square 54, let $x = 4$. Then $100\,(25 + x)$ is 100×29, or 2900; x^2 is 4^2, or 16. The answer is $2900 + 16$, or 2916.

To square 47, let $x = -3$. Then $100\,(25 + x)$ is 100×22, or 2200; x^2 is 3^2, or 9. The answer is $2200 + 9$, or 2209.

Whole numbers having a value near 25 may be squared with the help of the following relationship:

$$(25 + x)^2 = 100x + (25 - x)^2$$

To square 36, x is 11, and the answer is $100 \times 11 + (25 - 11)^2$, or $1100 + 196$, or 1296.

Oddly enough, this relationship works better, for most people, using numbers in the thirties than numbers in the twenties, because the average person does not know the square of numbers from 21 to 29, inclusive.

Any one of the above or similar rules affords excellent enrichment material for the entire class. The superior student should then be challenged to demonstrate why the rule works. For variation, the algebraic relationship may be presented first with the challenge to interpret it in a practical computational way.

Most of the simple algebraic relationships or identities have some value for mental computation and shortcuts. These should be pointed out from time to time.

The following type of "trick" is also subject to algebraic analysis.

(1) Write down your weight and double it.
(2) Add 10 to this product.
(3) Multiply the resulting sum by 50.
(4) Add your age to this product.
(5) Subtract 500 from the last sum.

The last two digits in the difference (step 5) are your age and the first two or three digits give your weight. Normally, you should ask an individual to perform only the first four steps. By subtracting 500 from the result of step 4, you should be able to state the weight and age almost instantly.

To explain why such a "trick" works, the able student should be able to use algebra as follows:

Let x be the number of pounds in the weight of the individual and y be the number of years in his age. The results of following the instructions of each step are: (1) $2x$; (2) $2x + 10$; (3) $100x + 500$; (4) $100x + 500 + y$; (5) $100x + y$. Thus, as long as the age is a one-or-two-place number, the last two digits of the result of step 5 will give the age and the first two or three digits will give the weight.

e. An Introduction to Conic Sections

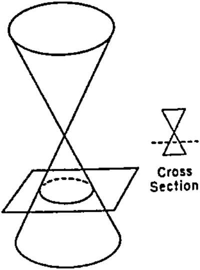
Why Study Conic Sections?

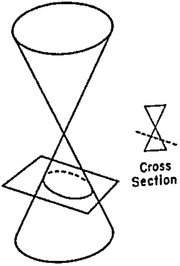
A non-technical study of the *conic sections* offers a worthwhile area of investigation for the superior student in the upper grades. There are two major reasons why the conic sections should be of interest to a student in the junior high school. First, the number of applications of these curves in everyday life is very large. It is worthwhile from a cultural point of view to recognize the many forms of these curves in our surroundings. Second, the conic sections are important from a mathematical point of view. They frequently are used to illustrate important mathematical principles in high school and college. Familiarity with the curves of this family and some of their properties can be of tremendous value for the student in his future study of mathematics.

Since the equations of these curves cannot be presented at the junior high school level, with the possible exception of the parabola in first year algebra, the study of the curves must be qualitative. Much can be learned about the shape, properties, and applications of each curve without knowing anything about its equation. The illustrations show a mathematical cone.

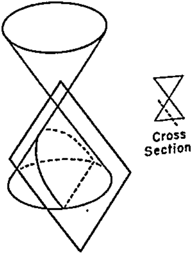
The ability to visualize space relationships is invaluable to engineers and mathematicians. The discovery of the relationship of the circle, parabola, hyperbola, and ellipse to a mathematical cone is an excellent exercise in visualizing space relationships. Every effort, however, should be made to provide drawings and models to help the student understand this relationship.

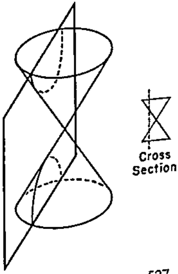
The vertex of a double cone, or a mathematical cone, is the point common to both cones, as the point O. The axis of such a cone is the line through the vertex perpendicular to a base. Any circular cross section of a mathematical cone may be considered to be a base of the cone. A *circle*, *ellipse*, and *parabola* may be obtained from a single cone. The *hyperbola* requires a double cone. These conic sections are formed by the intersection of a plane or flat surface with the cone. The relative position of the plane and the cone determines which curve is formed. It may help the student to imagine that the cone is being cut by a knife. If the blade





of the knife is held parallel to the base, a circle results as illustrated. If the blade is slanted just a little from the horizontal, an ellipse results as in the next drawing. If the cut is made parallel to one side of the cone, the curve is no longer closed and a parabola results. The second drawing below illustrates how the hyperbola may be formed.





The circle is the best known curve of the conic sections with which all students at the junior high school level are familiar. Its applications in the form of wheels, gears, and designs are understood by most people.

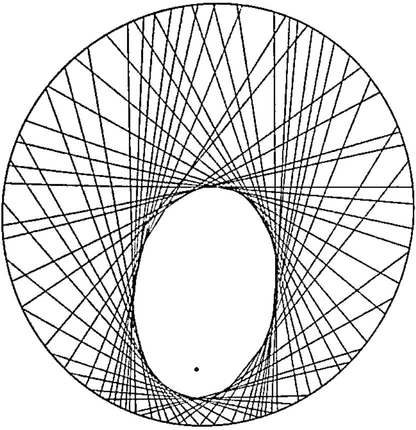
The parabola occurs in our daily lives far more frequently than most people realize. A ball thrown by two boys playing catch travels in the path of a parabola. The water from a garden hose in a non-vertical position travels in parabolic paths. The cross sections of the reflectors in automobile headlights and flashlights are parabolas. Some comets travel in a parabolic path. The giant cables on suspension bridges are in the shape of a parabola.

The earth travels about the sun in an elliptical path. Arches in bridges and buildings are often halves of ellipses. Gears are occasionally made in elliptical form to produce pulsating motion. Designers of patterns for fabrics and pottery frequently make use of the ellipse. One cross section of the famous whispering gallery in Washington is elliptical in shape. Mathematical properties of the ellipse are responsible for the unusual acoustical properties of this gallery.

The teacher should have the students fold paper so as to form an ellipse. Each student should have a piece of thin wax paper from which he can cut a circular piece about 6 inches in diameter. Then with a pencil he should mark on the paper a point about 1 inch from the circumference. Next, he should fold the paper so that the circular edge of the part folded will pass through the pencil point. Then he should make a crease in the paper. The fold should then be opened and a new crease near the first crease should be made in the same manner. This process should be repeated until at least 25 creases are distributed somewhat evenly about the circumference of the circle. The part of the circular paper bordered by the creases and containing the pencil mark will be approximately in the form of an ellipse.

There are not as many common applications of the hyperbola as there are for the circle, ellipse, and parabola. A hyperbola has technical applications in navigation and military situations.

Each student should be encouraged to keep a notebook or a scrapbook in which he keeps pictures of the conic sections and other applications of mathematics.

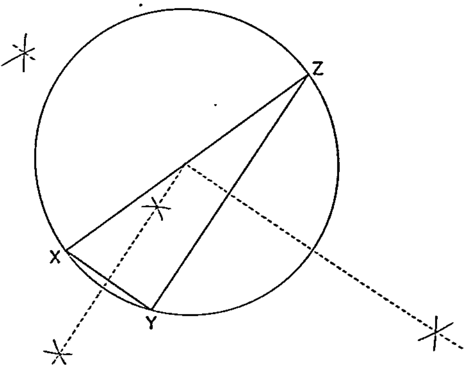


f. Geometric Constructions

The Most Familiar Constructions

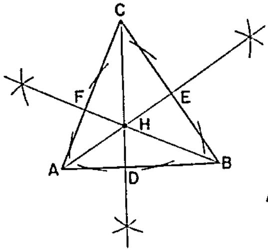
It is common practice to teach geometric constructions with straightedge and compasses in the junior high school. The most common include those given on page 379.

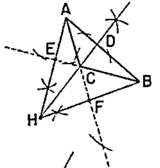
The more able students should be encouraged to find uses for these constructions. Many constructions required with proof in a high school course in plane geometry may be performed without proof at the junior high school level. The use of the perpendicular bisector of a line segment to circumscribe a circle about a triangle (place the circle such that the vertexes of the triangle are on the circle) is a good example. Such an activity is not only an introduction to one of the many fascinating relationships which

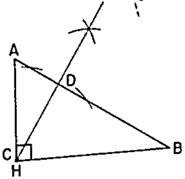


exist in geometry but it also serves as an incentive for accurate work. The circumscribed circle will pass through the three vertexes of the triangle only if the construction of the perpendicular bisectors of the sides is performed with considerable accuracy. In this sense, the problem is self checking, and the student should realize that the failure of the circle to pass through the three vertexes is an indication of inaccurate work. The equivalence of the construction just discussed with that of drawing a circle through any three points not in the same straight line should be mentioned. Note the application of this construction to the points X, Y, and Z above, and that only two perpendicular bisectors are necessary.

In the same manner, bisecting an angle may be used to inscribe a circle in a triangle. The three angle bisectors meet at a point which is the same distance from all three sides and therefore must be the center of the circle *tangent* to all three sides. This construction affords an excellent introduction to the meaning of tangency. This exercise is self checking in the same manner that the construction of a circumscribed circle is self checking.







An altitude of a triangle is determined by constructing a perpendicular from one vertex of a triangle to the side opposite this vertex (or this side extended). By carefully measuring this base and altitude and substituting the results in the formula $A = \frac{1}{2}bh$, it is possible to approximate the area of the triangle. If a perpendicular is then constructed from a different vertex of the same triangle to the side opposite this vertex (or this side extended), the area also can be determined from this new base and altitude. In a similar manner the area may also be determined by constructing a third altitude. In a *scalene* triangle, the three pairs of measurements will be different but the three areas will be the same, allowing for minor deviations due to inevitable errors in construction and measurement. This is an excellent exercise for combining accuracy of measurement with accuracy of construction. The student should perform this construction with acute, right, and obtuse triangles, as in the accompanying figures. It may then be noted that in each case the three altitudes of the same triangle pass through a common point. The altitudes are said to be *concurrent*.

A student showing interest or ability in geometry should be encouraged to read beyond his text in search of new facts and relationships. He should have access to at least one text in high school plane geometry. Much of the descriptive material in such a text is within his comprehension. A bright student, endowed with intellectual curiosity, browsing through a high school text might very well learn the following things:

(1) A median is a line from any vertex of a triangle to the midpoint of the opposite side.

(2) The three medians of any triangle pass through the same point (are concurrent).

(3) This point of concurrency is the center of gravity of the triangle.

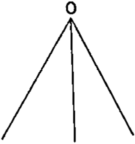
This student should be encouraged to cut a triangle from cardboard and construct its center of gravity. By sticking a pin through the point of concurrency of the medians and spinning the triangle, he can demonstrate that the triangle will stop at different positions in its rotation. By sticking the pin through any other point, he can demonstrate that then the triangle is no longer in balance and will always stop at the same position.

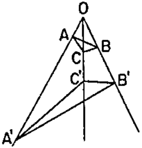
There are a number of theorems found in advanced geometry which may readily be performed and understood by students at an upper grade level. These theorems represent advanced work due to the methods of proof and maturity of students required for understanding the process. Frequently the constructions which illustrate these theorems are simple and interesting, as *Desargue's Theorem* and the *Nine Point Circle.*

Desargue's Theorem

The student needs only a ruler and pencil to demonstrate Desargue's Theorem (25 pp. 144–48). This theorem states that if two triangles are in *perspective,* their corresponding sides meet in three points that lie on the same straight line. These points are said to be *collinear.*

Two triangles are in perspective when the lines joining their corresponding vertexes pass through the same point. This is the basic idea of perspective used by commercial artists.

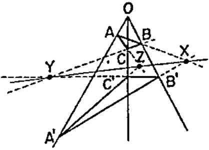


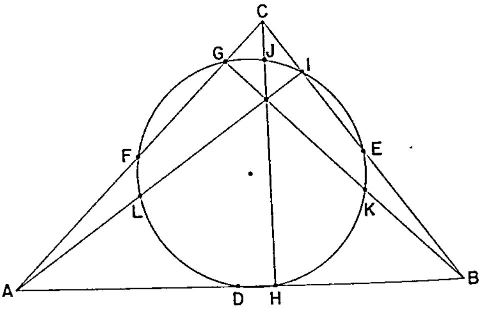


The first drawing above shows three lines passing through point O. In the next drawing, the two triangles are in perspective because their vertexes lie on lines intersecting at point O. A and A′, B and B′, C and C′ lie on the same line. This notation makes it easy to identify corresponding vertexes as well as corresponding sides. Since the corresponding sides lie between corresponding vertexes, the corresponding sides are:

AB—A′B′
BC—B′C′
AC—A′C′

The truth of the theorem is demonstrated in the figure below because the pairs of corresponding sides, as indicated above, meet in the points X, Y, and Z, respectively, which do lie on the same straight line. The theorem apparently breaks down when one or more pairs of corresponding sides are parallel. Mathematicians have an explanation for this situation but its proof is beyond the scope of junior high school mathematics. Consequently, students interested in this construction should be cautioned to avoid situations in which corresponding sides are parallel or almost parallel.





Nine Point Circle

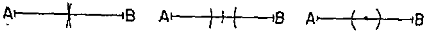
Probably the most famous circle in geometry is the *nine point circle.* In any triangle nine points (three sets of three points) lie on the same circle.

1. The three mid-points of the sides of the triangle are D, E, and F.

2. The three feet (points of intersection of altitudes with sides) of the altitudes are G, H, and I.

3. The three mid-points of the line segments joining the point of concurrency of the altitudes to the vertexes are J, K, and L.

It can be proved that the center of this circle is midway between the common point of the altitudes and the common point of the perpendicular bisectors of the sides (center of the circumscribed circle) Probably the easiest way to find the center of the



nine point circle is to construct the perpendicular bisectors of segments HD and EI and note that their intersection is the desired center. This is true because the perpendicular bisectors of two chords of a circle pass through the center of the circle. Refer to the drawing on page 383 again.

Although the mid-point of a line segment is desired, the perpendicular bisector through this point is not needed. The following construction is practical even though it is not theoretically exact. First, estimate one-half of the line segment. From each end of the line segment describe a short arc intersecting the line. If the estimation is correct, the two arcs cut the line in the same point which is the required mid-point. If the estimation is not correct it is easy to determine the center of the short line segment between the two arcs. The drawings show a situation in which the estimation was correct, in which the estimated distance was too small, and in which the estimated distance was too large. Unless the initial estimation is greatly in error, the center can be determined sufficiently accurately by this method.

The above method of determining the center of a line segment is valuable in constructing a nine point circle.[5] If the usual method of bisecting a line segment is used to bisect the six line segments in constructing a circle of this type (construction 1 on page 379), the work becomes so crowded that the desired points of intersection are not discernible.

Construction of Geometric Models

The construction of mathematical models can be an exciting and profitable experience for any student. This type of activity is an excellent means for giving gifted students additional opportunities to explore the nature of their interests and abilities. Extreme care should be taken, however, to insure that such

[5] For a modification of the above plan, see Satterly, John, "Another Approach to the Nine-point Circle," *The Mathematics Teacher*. 50 53–54.

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When the whole class participates, simpler as well as more difficult models may be constructed.

projects are not merely exercises in carpentry with no concomitant learning in mathematics. It also should be recognized that the construction of an original model may involve judgments and reasoning which are essentially mathematical even in situations in which no computation is involved.

An excellent problem in model construction is to make a cone having a given radius and altitude. Only the very superior student will be able to make such a model from a piece of paper without help from the teacher. The student should first discover that the cone is formed from a piece of paper cut in the form of a sector of a circle. To construct a cone with a given radius and altitude, it is necessary to determine the size of the vertex angle (angle A) and the length of the radius (AB) of the sector.

Consider the radius of the cone as 2 inches and the altitude as 3 inches, which are approximately the dimensions of a No. 202 can. It must be understood that the radius of the sector, AB, is the *slant height* of the cone. This unknown line segment is then

the hypotenuse of a right triangle with legs equal to the required radius and altitude of the cone as illustrated. It is necessary then to solve the equation, $x^2 = 2^2 + 3^2$. Hence $x = 3.6$ inches.

The length of arc BC is equal to the circumference of the circular base of the cone. The length of arc BC may be determined as follows:

$$C = 2\pi r$$
$$C = 2 \times 3.14 \times 2, \text{ or approximately } 12.6$$

The length of the arc BC is 12.6 inches.

The circumference of the entire circle from which the sector ABC has been cut may be determined in a similar manner:

$$C = 2\pi r$$
$$C = 2 \times 3.14 \times 3.6, \text{ or approximately } 22.6$$

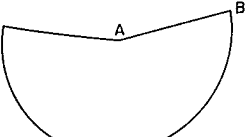
The circumference of the circle is 22.6 inches.

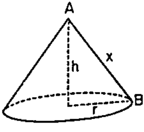
Arc BC is .558 or 55.8 per cent of the circumference of the circle from which sector ABC has been cut. From this it follows that angle A is 55.8% of 360° or approximately 201°. For the student familiar with proportions, the problem may be solved in the following manner:

$$\frac{12.6}{22.6} = \frac{x}{360} \qquad 22.6x = 4536$$
$$x = 201$$

The angle intercepted by the arc BC is approximately 201°.

After these computations are completed, the student must then choose a suitable material. He must decide how to join the edges. If the edges are to be overlapping, provision for this feature must be made. When made from material which is water resistant, this cone may be used with a No. 202 can to demonstrate the relationship between the volume of a cone and the volume of a cylinder when these figures have like dimensions.





The Eighteenth Yearbook (34) suggests many models that are suited to the junior high school level. The department entitled "Devices for the Classroom" in *The Mathematics Teacher* frequently suggests models appropriate for construction by junior high school students.

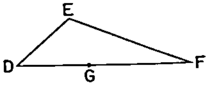
g. Miscellaneous Means of Enrichment

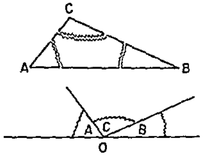
Paper Folding

Paper folding can be both interesting and educational. It is easy to form the perpendicular bisector of a line segment by paper folding. Many geometric theorems may be demonstrated by paper folding. The folding may be done on a thin translucent paper, such as wax paper, or with geometric figures cut from ordinary paper.

From a piece of colored writing paper, cut a triangle, and call it ABC. Place vertex A on vertex B. Then form a crease so half of segment AB folds on the other half and the folded triangle is perfectly flat. This crease is the perpendicular bisector of line segment AB. The creases representing the perpendicular bisectors of AC and BC may be formed in a similar manner. This is probably the simplest and most effective way of convincing a student that the perpendicular bisectors of the three sides of a triangle are concurrent.

From paper cut another triangle, such as triangle DEF. Let G be a point on line segment DF so that DE is equal to DG. Now place point E on point G and form a crease so DE falls along DG and the folded triangle is flat. This crease must pass through D and thus forms the bisector of angle D. The creases representing the bisectors of angles E and F may be formed in a similar manner. The demonstration shows effectively that the three angle bisectors pass through a common point.





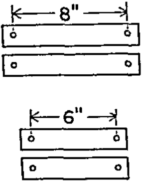
With a little ingenuity, this method may be used to demonstrate that the three medians and the three altitudes of a triangle are concurrent. With triangles cut from paper, the three altitudes cannot be demonstrated for an obtuse triangle since the common point of the altitudes lies outside the triangle.

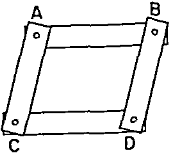
The fact that the sum of the three angles of any triangle is equal to 180° may be demonstrated with the aid of some paper tearing. Cut any triangle from ordinary paper. Tear the triangle into pieces as indicated and place the pieces as illustrated in the next drawing. The sum of the three angles obtained in this manner may then be checked with a ruler or any other straight-edge to verify experimentally that the sum of the three angles of a triangle is a straight angle or 180°.

Other references to paper folding may be found in *Winter Nights Entertainments* (2), *Geometric Exercises in Paper Folding* (39), and *Geometric Tools* (55). The latter two are probably of more value to teachers than to most students in the junior high school.

Linkages and Moving Models

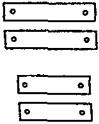
Several eyelet punches with a generous supply of #2 and #3 eyelets should be standard equipment in every mathematics classroom for the upper grades. Such a punch used with 14 ply cardboard will make it possible to construct a large variety of moving models and *linkages*. A linkage is a device with moving parts that may be used to construct geometric figures or demonstrate their properties.

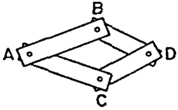


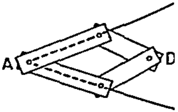


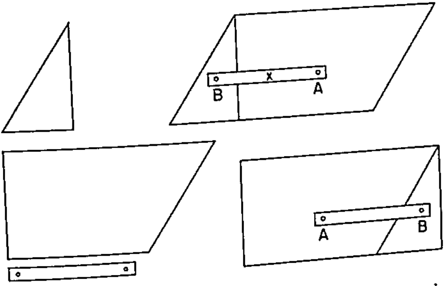
A simple model to construct is a parallel ruler. The four parts required are shown. The next drawing shows a parallel ruler assembled with eyelets at A, B, C, and D. The geometric principle, *that any quadrilateral with opposite sides equal must be a parallelogram*, applies to a parallel ruler. AB, CD, AC, and BD are strips of heavy cardboard. Since AB = CD and AC = BD by construction, the edges AB and CD must be parallel.

An angle bisector is another simple project to construct. The four pieces of heavy cardboard to be used for this project are shown first. Then the device assembled is shown. To bisect an angle with this device, place the instrument so that the center of the eyelet at A lies directly over the vertex of the angle to be bisected, while the centers of the eyelets B and C lie on each side of the angle as demonstrated. By placing a pencil through the eyelet at D, a point on the angle bisector is located. A straight line segment connecting this point to the vertex is the angle bisector.









The pieces of heavy cardboard for a model which may be used to demonstrate how a parallelogram may be transformed into a rectangle with the same area, base, and altitude are shown above. The drawing, upper right, shows how the model looks as a parallelogram, while the drawing below it shows how it may be rearranged easily and quickly to form the desired rectangle. The only problem in the assembly of these three parts is that of determining where to locate the eyelet at A. Take the following steps:

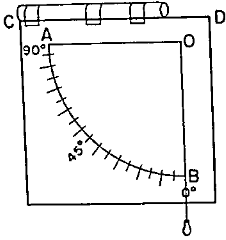
1. Attach one end of an arm, as *x*, to the triangle at B with an eyelet. Then punch a hole near the other end of the arm where the other eyelet is to be used.

2. Place the triangle beside the quadrilateral as shown upper right and place a sharpened pencil through the hole and describe an arc on the quadrilateral.

3. Place the triangle in the position shown lower right and again place a pencil in the hole at the other end of the arm and describe an arc.

4. The point of intersection of the arcs formed in steps 2 and 3 is the point A where the arm should be joined to the quadrilateral (trapezoid) by the second eyelet. A convenient length for the arm can be found by experimentation.

The gifted student should be challenged to discover how to locate point A without help.



A quadrant, like the one shown, may be made as follows:

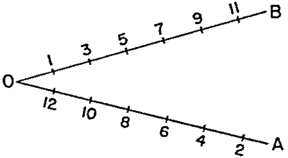
1. Obtain a 12-inch square of 14 ply cardboard.
2. Select a straight edge, as CD shown.
3. Choose point O about one inch from point D.
4. Draw the 10-inch line OA parallel to CD and the 10-inch line OB perpendicular to CD.
5. With O as the center, draw arc AB (10-inch radius).
6. Use a protractor to measure the arc AB into divisions of at least 5° each.
7. With the aid of an eyelet punch, attach a weighted string about 14 inches long to point O.
8. Use tape to attach a soda straw (or metal tube about the size of a soda straw) to side CD for an eyepiece.

See page 355 for suggested uses for such a quadrant.

The Mathematics Teacher usually has one of its sections devoted to models appropriate for construction by students and teachers. Many of these constructions are suitable to some phase of junior high school work. The issues of October 1951 and April and May 1952 have suggestions that are of value to junior high school students. *The Eighteenth Yearbook* (34) should be in every junior high school library as it has many suggestions for making models.

Commercial plastics which are very pliable in the formative stage but after a short time set to form rather rigid models may be investigated as materials for models.

Opportunity should be given to students to use and demonstrate devices which they have constructed

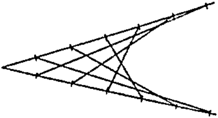


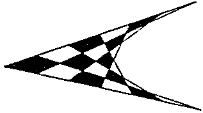
Curve Stitching

In recent years, *curve stitching* has been a rather popular activity over a wide range of grade levels. A simple example of curve stitching can be made by following the drawing above. Fasten two heavy strips of cardboard at a point so as to form an acute angle. Then punch eyelets at equal intervals on each of the sides of the angle. Number eyelets on one side as even numbers and on the other side as odd numbers in the order shown. Next, thread a needle, preferably with colored thread, and tie a knot for an anchor. Insert the needle from the underside of point 1, pull the thread taut, and then proceed to point 2 and insert the needle again. Thus, the needle will be inserted from the underside for all odd numbers and from the upper side for all even numbers. When all the numbers have been used consecutively in this manner, the result will be a series of straight lines (threads) outlining a parabola. Further information on curve stitching may be obtained from a brief article in the *Eighteenth Year Book* (34) and in *A Rhythmic Approach to Mathematics* (47).

Objections to curve stitching have been expressed because it sometimes is done without reference to the mathematics involved. In other instances, such as the example given, the mathematics involved is beyond the level of a junior high school student. However, the activity gives the student one more opportunity to recognize a parabola and see one more way in which it may be formed.

The esthetic value of curve stitching should not be overlooked. If not overemphasized, curve stitching can be a useful activity. On the other hand, an activity of this kind should not eliminate projects which have more direct mathematical implications.





Mathematics in Design

The drawing of designs, original or otherwise, is a useful activity for junior high school students. This activity provides an excellent correlation between art and mathematics, and promotes skill in the use of essential drawing instruments. A pattern directly related to the curve stitching example of the previous section is illustrated in the sequence of drawings shown above.

Baravalle suggests a wide variety of such drawings in *Geometry at the Junior High School Grades* (8) and to a lesser extent in the *Eighteenth Yearbook* (34). Again, care should be taken not to overemphasize such a program to the detriment of basic mathematics.

Slide Rule

The slide rule is a very useful instrument to emphasize certain features of our number system and for giving practice in estimating answers to many computational problems.

It is not common practice to introduce the slide rule to an entire class at the junior high school level, but it is worthwhile to have a number of slide rules with instruction booklets available. The superior students should be encouraged to learn how to use a slide rule from such booklets. The teacher should help the students clarify points of difficulty which may be encountered. If these students demonstrate a certain proficiency in the use of the slide rule, they should be permitted to use them to do computational work in class exercises and tests. If a student uses a slide rule to find the product of two numbers, the teacher should know that the answer was obtained by use of a slide rule in order to understand that the answer usually will be a close approximation of the true value instead of the exact value.

Field Work in Mathematics

Many of the fundamental ideas of surveying and similar outdoor work are within the range of the majority of junior high school students. There is also much additional worthwhile material for the gifted student.

All students should learn to make simple measurements in inches, feet, yards, and some metric units such as meters or centimeters. All students should make a simple scaled diagram of the classroom, including cabinets and tables. The more able and interested students could make a map of the school grounds, including the building, using an inexpensive or home made transit for obtaining needed measurements.

Students should be encouraged to make simple instruments which can be used in indirect measurements. References No. 35 and 43 contain many suggestions for making and using instruments in indirect measurements.

Bulletin Board and "Math" Table

The bulletin board is one of the oldest and most useful of all enrichment devices. Teacher and students should collect a file of interesting material from newspapers, magazines, and any other source for display at appropriate times. The work involved in keeping the bulletin board up to date should be done by the students. They should be encouraged and required to bring in material pertaining to mathematics for the bulletin board.

Committees may be appointed to be in charge for periods of a week. In any case, the bulletin board should reflect the many aspects of mathematics and its applications to everyday life. The humorous aspect, as reflected in many cartoons, should not be ignored.

In addition to a bulletin board, a mathematics table is very desirable. The table for demonstration purposes as recommended in Chapter 4 may be used at various times for displaying materials. In a room devoted only to the teaching of mathematics, such a table should be used to display models, new books, pamphlets, puzzles, and other material of interest which cannot

be displayed on a bulletin board. When the room is used for all or most of the subjects, mathematics displays and materials will be alternated with displays for other subjects on this table. This is in addition to the use of the table as a platform for demonstrating number relationships on a flannel board and similar activities.

Pamphlets and display material from business and industry should not be overlooked. The following material may be obtained by writing to the organization indicated:

Athletic Field and Court Diagrams
Lowe and Campbell, 225 N. Wabash Avenue, Chicago, Illinois.
The Story of Figures and Fascinating Figure Puzzles.
Burroughs Adding Machine Co , 6071 Second Blvd., Detroit 32, Michigan.
Chart of Decimal Equivalents
Monroe Calculating Machine Co , Orange, New Jersey
How It Works.
Westinghouse Electric Corporation, School Service
306 Fourth Avenue, Pittsburgh, Pennsylvania

"*Math*" *Clubs*

Where feasible, a mathematics club is very desirable. This will generally attract the more able and interested students. The topics already discussed in this chapter make an excellent guide for the type of material which can be used for a mathematics club.

"Math" clubs provide an opportunity for extending learnings from the mathematics classroom.

Regional High School, Springfield, New Jersey

Every effort should be made to contact the faculty in other curriculum areas for suggestions on how mathematics may be of value in these areas.

Mathematics contests are held in many sections of this country for various grade levels. Students interested in taking such examinations must be provided with opportunities for an enriched curriculum.

Teachers should constantly keep in contact with newly released courses of study and current periodicals for new areas of enrichment. Films and visual aids may frequently help to enrich the mathematics curriculum for a few gifted students or for an entire class.

Changes in Modern Mathematics

A commission has been set up under the auspices of the College Entrance Examination Board to study the possibilities of introducing ideas of modern mathematics into the elementary curriculum. (See pages 454–455.) *Set theory* is one of the topics that the commission recommends for possible inclusion in the secondary curriculum in mathematics. The concept of sets is less than 100 years old but most branches of modern mathematics treat of sets. The concept is a major unifying idea in mathematics.

From an elementary point of view, a set is a collection of elements. These elements may be numbers, points, geometric figures, or similar mathematical quantities.

The interested reader is referred to the Twenty-third Yearbook of the National Council of Teachers of Mathematics, *Insights into Modern Mathematics.* It gives a discussion of sets.

Some schools introduce inequalities and absolute notation at the ninth year level. The University of Illinois experimental program, subsidized by the Carnegie Foundation, has introduced these ideas in first year algebra with considerable success. The work described by Beberman and Meserve[6] is representative of the kind of work that can be accomplished by superior students in elementary mathematics.

[6] Beberman, M. and Meserve, B. E. "Graphing in Elementary Algebra," *The Mathematics Teacher.* 49:260–266.

Questions, Problems, and Topics for Discussion

1. What are some of the characteristics of a gifted student?
2. Outline a plan for providing enrichment of the curriculum for several gifted students in a class of 30 students.
3. Give an outline for a unit of work suitable to a gifted student in the junior high school.
4. Find a topic not mentioned in this chapter suitable for enrichment purposes.
5. How should a mathematics textbook be used in an enriched program?
6. Evaluate some particular textbook in mathematics with respect to its provisions for enrichment.
7. Make a magic square with seven cells on one side.
8. Find an algebraic identity not mentioned in this chapter which may be used for computational shortcuts.
9. Find a newspaper or magazine article involving the use or discussion of a formula.
10. Collect a series of photographs from newspapers and magazines which illustrate various geometric figures.
11. Suggest ways in which a student's notebook might be used for enrichment.
12. What nine points are associated with the nine point circle?
13. From an appropriate book in the bibliography, select a problem requiring mathematical methods for solution without requiring any computation.
14. Make a list of topics appropriate for group discussions in a junior high school math club.

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* Books marked with an asterisk are too advanced for almost all junior high school students, but have background material that may be of interest to many teachers.

** This supplement to the *Chicago School Journal* contains bibliographies, lists of films, sources of free and inexpensive materials, and other information of interest to the alert teacher.

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